www.utm.my

CHAPTER 8

MAGNETOSTATIC FIELD

(MAGNETIC FORCE, MAGNETIC MATERIAL AND INDUCTANCE)

8.1 FORCE ON A MOVING POINT CHARGE

8.2 FORCE ON A FILAMENTARY CURRENT

8.3 FORCE BETWEEN TWO FILAMENTARY CURRENT

8.4 MAGNETIC MATERIAL

8.5 MAGNETIC BOUNDARY CONDITIONS

8.6 SELF INDUCTANCE AND MUTUAL INDUCTANCE

8.7 MAGNETIC ENERGY DENSITY

8.1 FORCE ON A MOVING POINT CHARGE

Force in electric field:

$$\overline{F}_e = Q\overline{E}$$

Force in magnetic field:

$$\overline{F}_m = Q\overline{U} \times \overline{B}$$

Total force:

$$\overline{F} = \overline{F}_e + \overline{F}_m \quad \textbf{or} \quad \overline{F} = Q\left(\overline{E} + \overline{U} \times \overline{B}\right)$$

Also known as Lorentz force equation.

Force on charge in the influence of fields:

Charge Condition	$\overline{E}$ Field	$\overline{B}$ Field	Combination $\overline{E}$ and $\overline{B}$
Stationary	$Q\overline{E}$	-	$Q\,\overline{E}$
Moving	$Q\overline{E}$	$Q\overline{U} \times \overline{B}$	$Q\left(\overline{E} + \overline{U} \times \overline{B}\right)$

8.2 FORCE ON A FILAMENTARY CURRENT

The force on a differential current element , $I\,\overline{dl}$ due to the uniform magnetic field, $\overline{B}$:

$$d\overline{F} = I d\overline{l} \times \overline{B}$$

$$\overline{F} = \oint I\,\overline{dl} \times \overline{B} = -I \oint \overline{B} \times \overline{dl}$$

$$\overline{F} = -I\overline{B} \times \oint d\overline{l} = 0$$

It is shown that the net force for any close current loop in the uniform magnetic field is zero.

Ex. 8.1: **A semi-circle conductor carrying current I, is located in plane xy as shown in Fig. 8.1. The conductor is under the influence of uniform magnetic field, $\overline{B} = \hat{y}B_0$. Find:**

(a) Force on a straight part of the conductor.

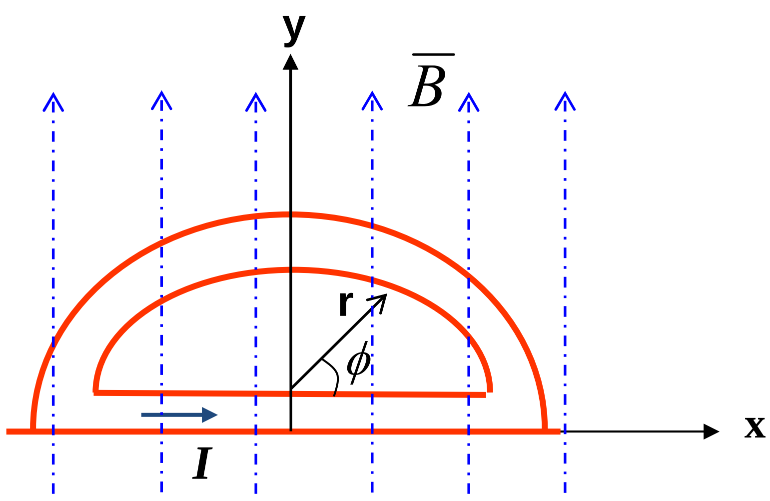
(b) Force on a curve part of the conductor.

Solution:

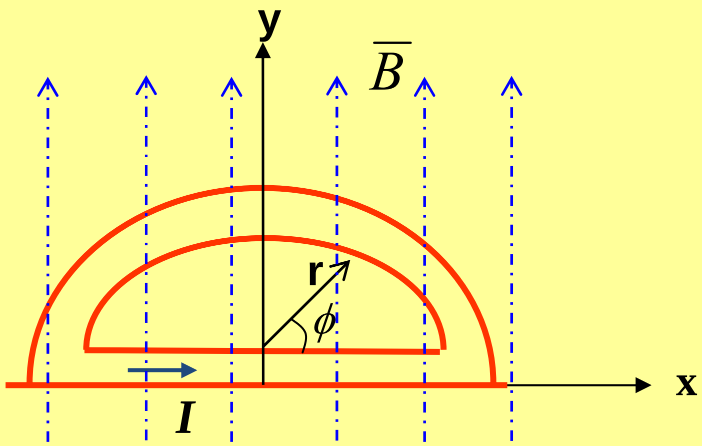
(a) The straight part length = 2r.
Current flows in the x direction.

$$\overline{F} = \int I\, \overline{dl} \times \overline{B}$$

$$\overline{F}_1 = \hat{x}(2Ir) \times \hat{y}B_0 = \hat{z}2IrB_0 \text{ (N)}$$



(b) For curve part, $\overline{dl} \times \overline{B}$ will be in the –ve z direction and the magnitude is proportional to sin ϕ

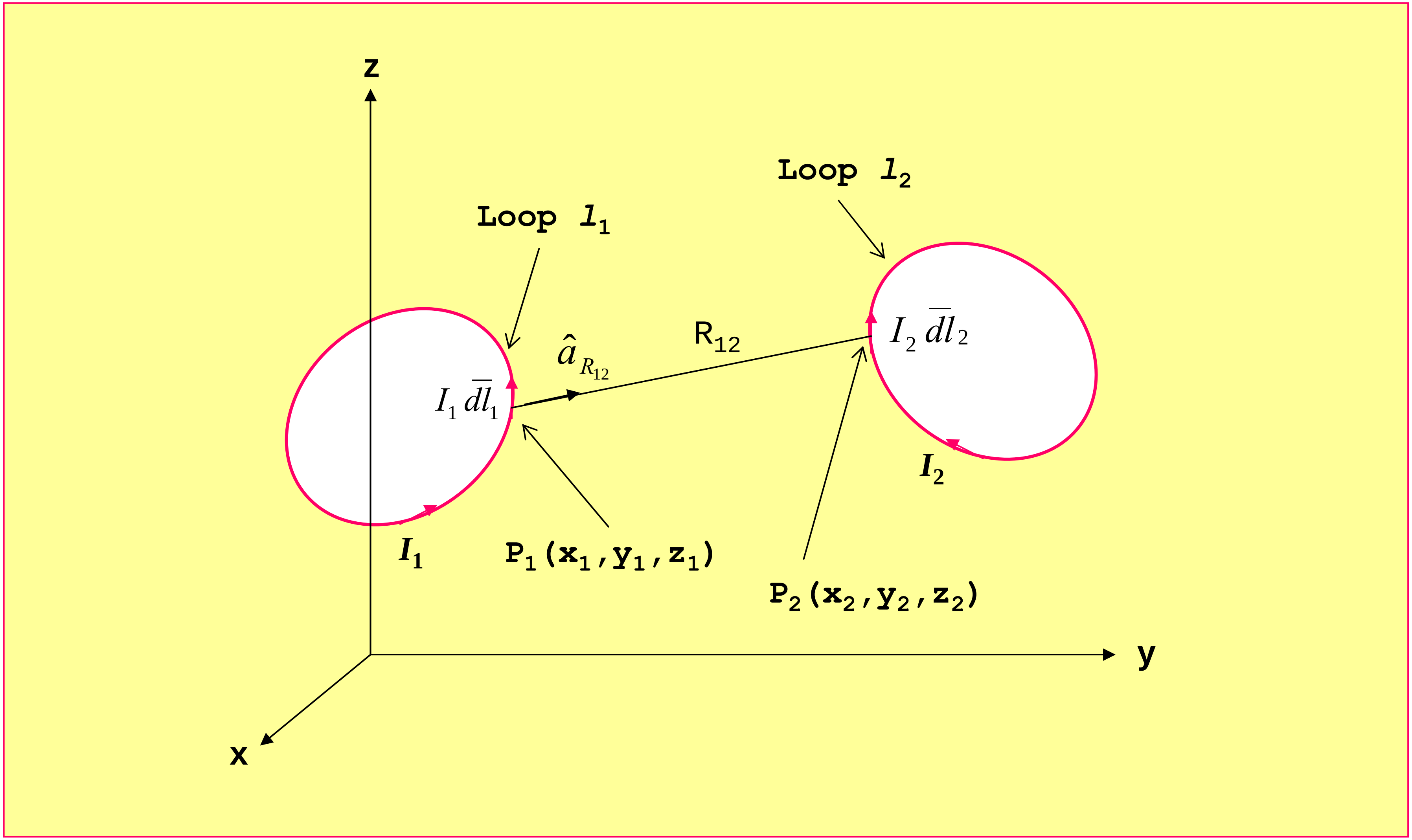


$$\overline{F}_2 = I \int_{\phi=0}^{\pi} d\overline{l} \times \overline{B}$$

$$= -\hat{z} I \int_{\phi=0}^{\pi} r B_0 \sin \phi d\phi = -\hat{z} 2 I r B_0 \text{ (N)}$$

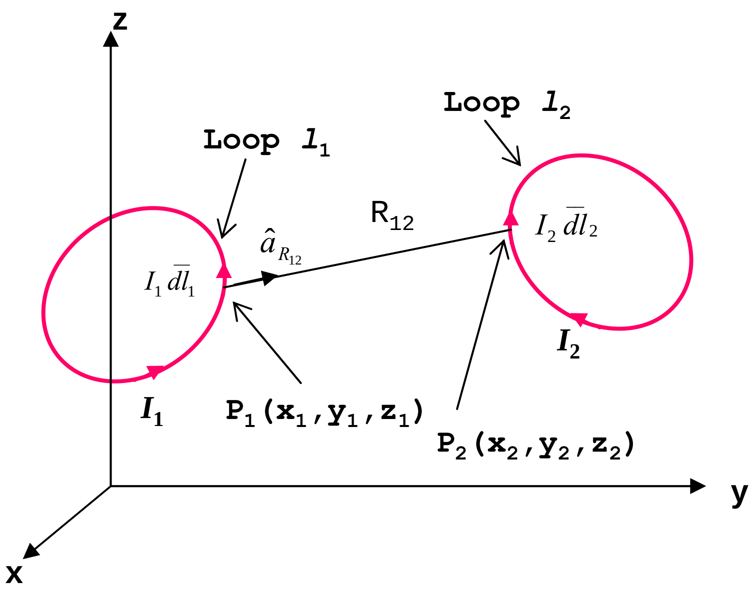
Hence, it is observed that $\overline{F}_2 = -\overline{F}_1$ and it is shown that the net force on a close loop is zero.

8.3 FORCE BETWEEN TWO FILAMENTARY CURRENT



We have :

$$d\bar{F} = I d\bar{l} \times \bar{B} \quad \textbf{(N)}$$



The magnetic field at point P_2 due to the filamentary current $I_1 dl_1$:

$$d\bar{H}_2 = \frac{I_1 d\bar{l}_1 \times \hat{a}_{R_{12}}}{4\pi R_{12}^{\;2}} \quad \textbf{(A/m)}$$

$$d\left(d\bar{F}_2\right) = I_2 d\bar{l}_2 \times \frac{\mu_o I_1 d\bar{l}_1 \times \hat{a}_{R_{12}}}{4\pi R_{12}^{\;2}}$$

$$\left(d\bar{F}_2\right) = I_2 d\bar{l}_2 \times \oint_{l_1} \frac{\mu_o I_1 d\bar{l}_1 \times \hat{a}_{R_{12}}}{4\pi R_{12}^{\;2}} = I_2 d\bar{l}_2 \times \bar{B}_2$$

where $d\bar{F}_2$ is the force at $I_2 dl_2$ and due to the magnetic field of loop l_1

Integrate:

$$\overline{F}_2 = \oint_{l_2} I_2 dl_2 \ \text{x} \ \oint_{l_1} \left[\frac{\mu_o I_1 d\overline{l}_1 \ \text{x} \ \hat{a}_{R_{12}}}{4\pi R_{12}^{\ 2}} \right]$$

$$\overline{F}_2 = \frac{\mu_o I_1 I_2}{4\pi} \oint_{l_2} \left[\oint_{l_1} \frac{\left(\hat{a}_{R_{12}} \ \text{x} \ d\overline{l}_1 \right)}{R_{12}^{\ 2}} \right] \text{x} \ d\overline{l}_2$$

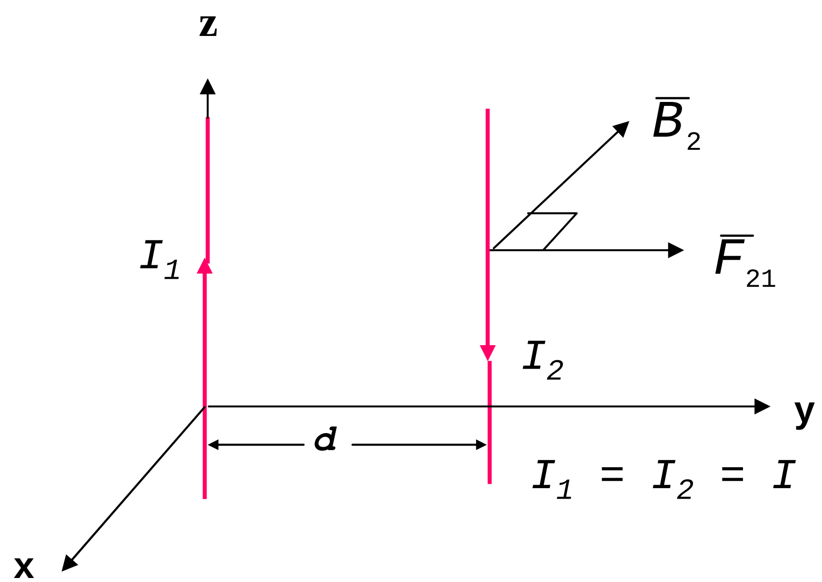
For surface current :

$$\overline{F}_2 = \int_{s} \overline{J}_{s2} \times \overline{B}_2 ds$$

For volume current :

$$\overline{F}_2 = \int_{v} \overline{J}_2 \times \overline{B}_2 dv$$

Ex. 8.2: Find force per meter between two parallel infinite conductor carrying current, *I* Ampere in opposite direction and separated at a distance d meter.



Solution:

$\overline{B}_2$ **at position conductor 2**

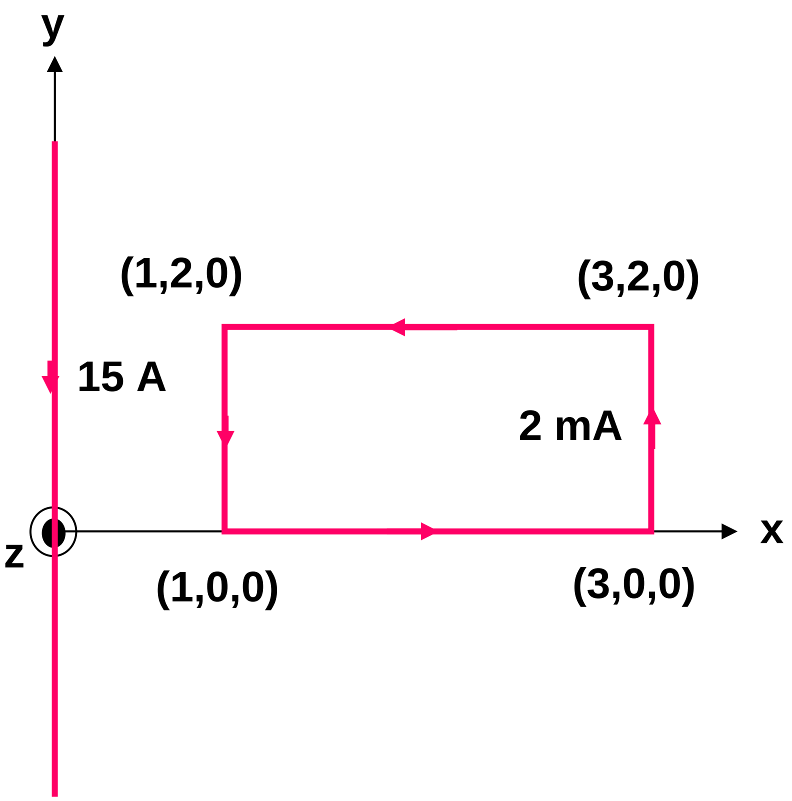
$$\overline{B}_2 = \mu_0 \overline{H}_2 = \mu_0 \frac{\hat{\phi} I_1}{2\pi r_c} = \frac{-\hat{x}\mu_0 I_1}{2\pi d}$$

Hence:

$$\overline{F}_2 = \int_0^1 I_2 d\overline{\ell}_2 \times \left(\frac{-\hat{x}\mu_0 I_1}{2\pi d} \right) = \int_0^1 I_2(-\hat{z}dz) \times \left(\frac{-\hat{x}\mu_0 I_1}{2\pi d} \right)$$

$$= \hat{y}\mu_0 \frac{I_1 I_2}{2\pi d} = \hat{y}\frac{\mu_0 I^2}{2\pi d} \qquad \text{(N/m)}$$

Ex. 8.3: A square conductor current loop is located in z = 0 plane with the edge given by coordinate (1,0,0), (1,2,0), (3,0,0) and (3,2,0) carrying a current of 2 mA in anti clockwise direction. A filamentary current carrying conductor of infinite length along the y axis carrying a current of 15 A in the –y direction. Find the force on the square loop.



Solution:

Field created in the square loop due to filamentary current :

$$\overline{H} = \frac{I}{2\pi x}\hat{z} = \frac{15}{2\pi x}\hat{z} \text{ A/m}$$

$$\hat{\phi} = \hat{a}_l \times \hat{a}_R = -\hat{y} \times \hat{x} = \hat{z}$$

$$\therefore \overline{B} = \mu_0 \overline{H} = 4\pi \times 10^{-7}\,\overline{H} = \frac{3\times 10^{-6}}{x}\hat{z} \text{ T}$$

Hence:

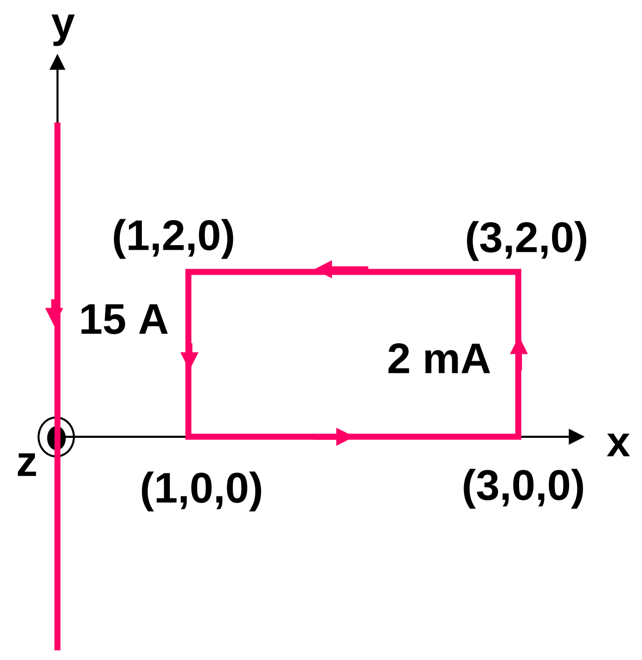
$$\overline{F} = \oint I\,\overline{dl} \times \overline{B} = -I \oint \overline{B} \times \overline{dl}$$

$$\overline{F} = -2\times10^{-3}\times3\times10^{-6}\left[\int_{x=1}^{3}\frac{\hat{z}}{x}\times dx\hat{x} + \int_{y=0}^{2}\frac{\hat{z}}{3}\times dy\hat{y} + \int_{x=3}^{1}\frac{\hat{z}}{x}\times dx\hat{x} + \int_{y=2}^{0}\frac{\hat{z}}{1}\times dy\hat{y}\right]$$

$$\overline{F} = -6\times10^{-9}\left[\ln x\Big|_1^3\,\hat{y} + \frac{1}{3}\,y\Big|_0^2\left(-\hat{x}\right) + \ln x\Big|_3^1\,\hat{y} + y\Big|_2^0\left(-\hat{x}\right)\right]$$

$$= -6\times10^{-9}\left[(\ln 3)\hat{y} - \frac{2}{3}\hat{x} + \left(\ln\frac{1}{3}\right)\hat{y} + 2\hat{x}\right]$$

$$= -8\hat{x}\ \text{ nN}$$



8.4 MAGNETIC MATERIAL

The prominent characteristic of magnetic material is magnetic polarization – the alignment of its magnetic dipoles when a magnetic field is applied.

Through the alignment, the magnetic fields of the dipoles will combine with the applied magnetic field.

The resultant magnetic field will be increased.

8.4.1 MAGNETIC POLARIZATION (MAGNETIZATION)

Magnetic dipoles were the results of three sources of magnetic moments that produced magnetic dipole moments : (i) the orbiting electron about the nucleus (ii) the electron spin and (iii) the nucleus spin.

The effect of magnetic dipole moment will produce bound current or magnetization current.

Magnetic dipole moment in microscopic view is given by :

$$\overline{dm} = I\,\overline{ds} \quad \mathbf{Am^2}$$

where $\overline{dm}$ is magnetic dipole moment in discrete and I is the bound current.

In macroscopic view, magnetic dipole moment per unit volume can be written as:

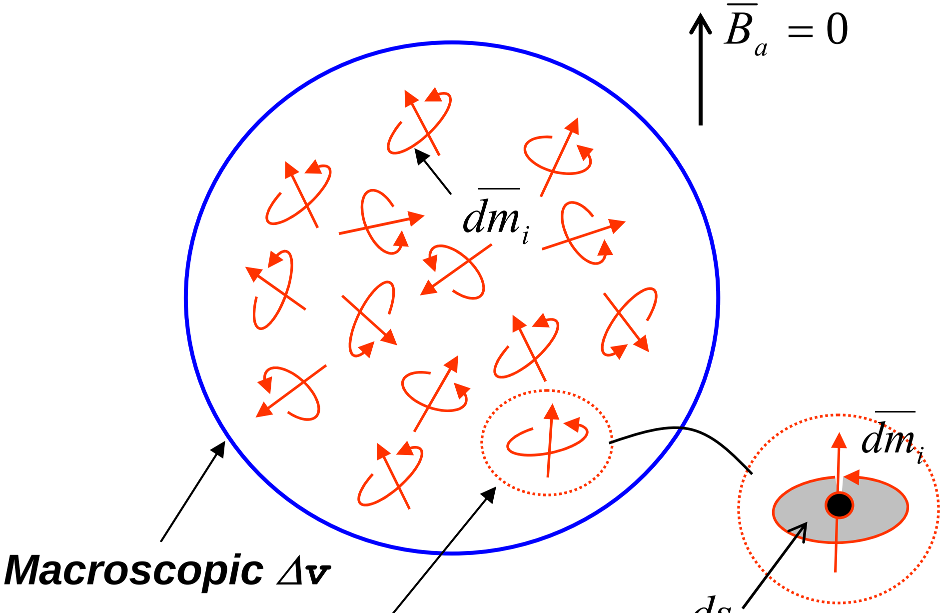
$$\overline{M} \cong \lim_{\Delta v \to 0}\left[\frac{1}{\Delta v}\sum_{i=1}^{n\Delta v}\overline{dm}_i\right] \quad \mathbf{A/m}$$

where $\overline{M}$ is a magnetization and n is the volume dipole density when Δv -> 0.

If the dipole moments become totally aligned :

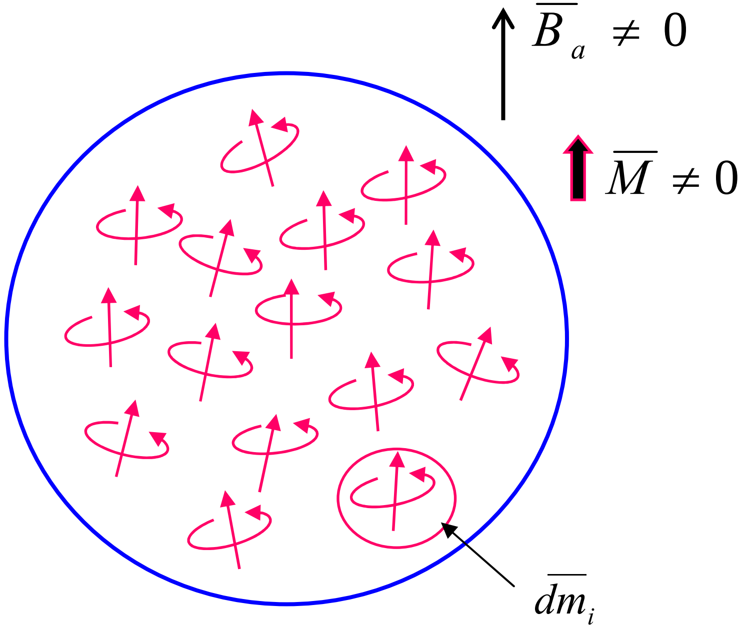
$$\overline{M} = n\,\overline{dm} = nI\,\overline{ds} \quad \mathbf{Am^{-1}}$$

Magnetic dipole moments in a magnetic material



$\overline{dm}_i \neq 0$

$\overline{M} = 0$



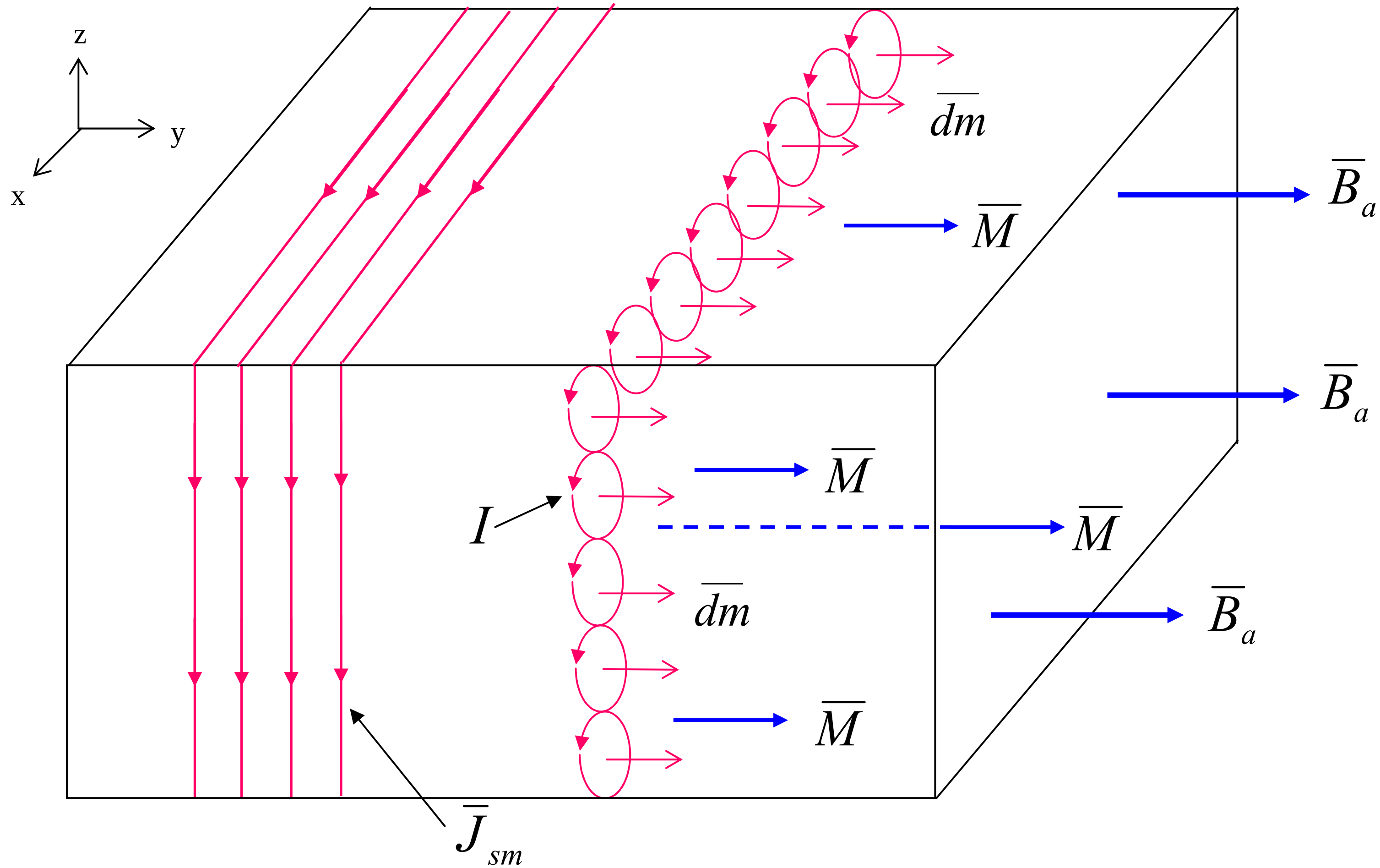
$\overline{dm}_i \neq 0$

$\overline{M} \neq 0$

$\overline{dm}'s$ ***tend to align themselves***

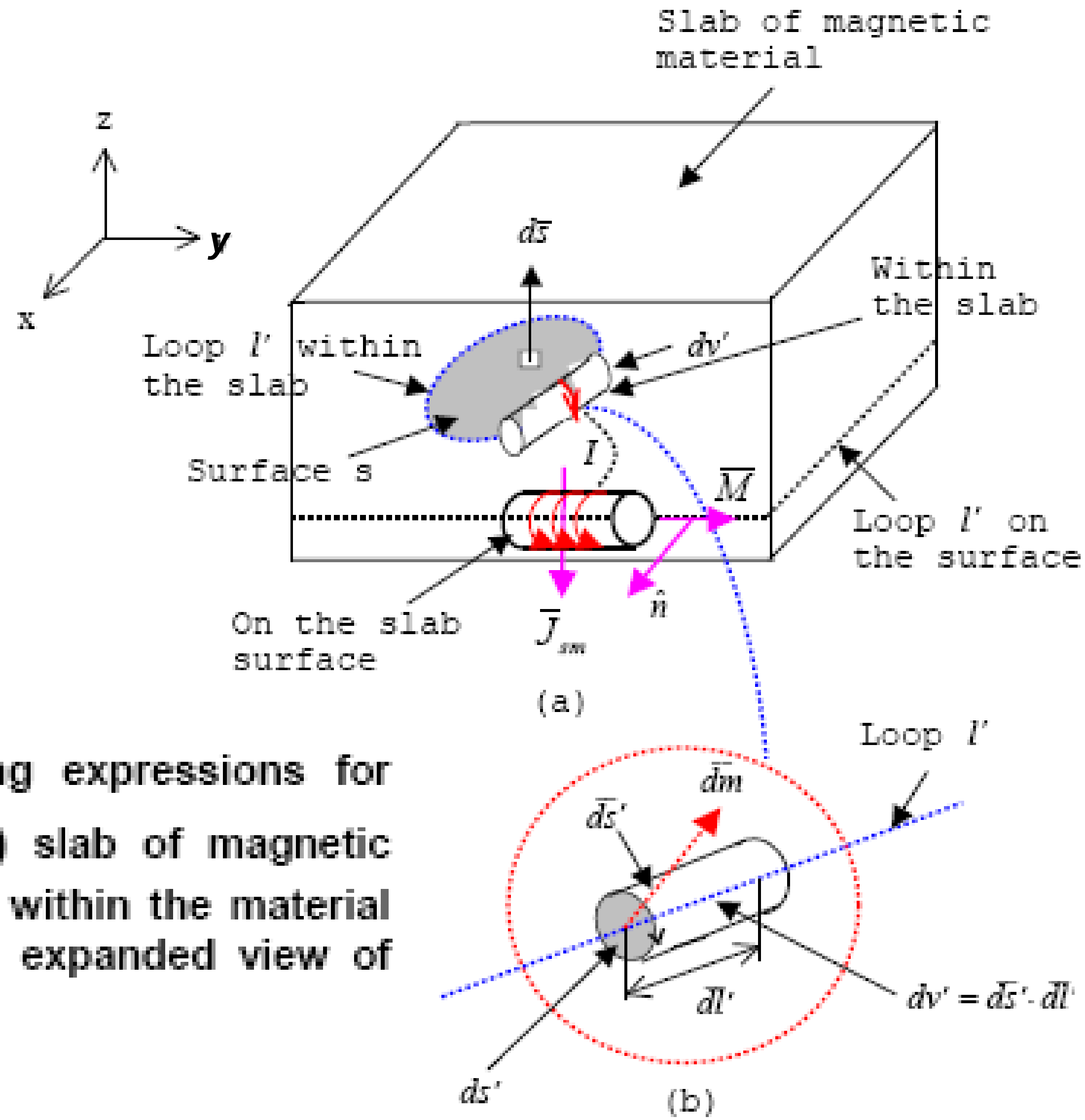
8.4.2 BOUND MAGNETIZATION CURRENT DENSITIES

$\bar{J}_{sm}$ and $\bar{J}_m$



Alignment of $\overline{dm}'s$ within a magnetic material under uniform $\bar{B}_a$ conditions to form a non zero $\bar{J}_{sm}$ on the slab surfaces, and a $\bar{J}_m = 0$ within the material.

8.4.3 TO FIND $\bar{J}_{sm}$ and $\bar{J}_m$



Graphical display for finding expressions for $\bar{J}_{sm}$ Am^{-1} and $\bar{J}_m$ Am^{-2}: (a) slab of magnetic material with closed loop l' within the material and on the slab surface (b) expanded view of dv' about the loop l'.

Bound magnetization current :

$$dI_m = Indv'$$

$$dI_m = I(n\,\overline{ds}'\cdot\overline{dl}') = (nI\,\overline{ds}')\cdot(\overline{dl}')$$

We have:

$$\overline{M} = n\,\overline{dm} = nI\,\overline{ds} \quad \mathbf{Am^{-1}}$$

Hence:

$$dI_m = \overline{M}\cdot\overline{dl}'$$

$$I_m = \oint \overline{M}\cdot\overline{dl}'$$ ***through the loop l'***

$$I_m = \int_s \overline{J}_m \cdot \overline{ds}$$ ***on the surface bound by the loop l'***

Using Stoke's Theorem:

$$I_m = \int_s \overline{J}_m \cdot \overline{ds} = \oint_l \overline{M} \cdot \overline{dl'} = \int_s \nabla \times \overline{M} \cdot \overline{ds}$$

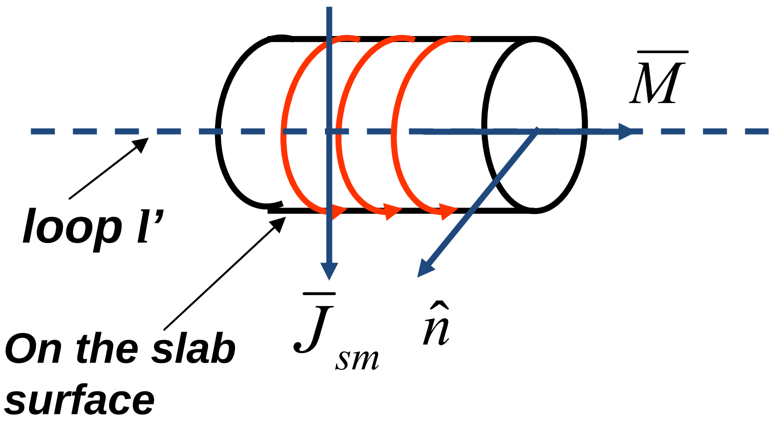
Hence:

$$\overline{J}_m = \nabla \times \overline{M} \quad (Am^{-2})$$

is the bound magnetization current density within the magnetic material.

And to find J_{sm} :

From the diagram : $dI_m = M_{tan}\, dl'$



$$\therefore M_{\tan} = \frac{dI_m}{dl'} \cong J_{sm}$$

$$\overline{J}_{sm} = \overline{M} \times \hat{n} \quad (Am^{-1})$$

is the surface bound magnetization current density

8.4.4 EFFECT OF MAGNETIZATION ON MAGNETIC FIELDS

Due to magnetization in a material, we have seen the formation of bound magnetization and surface bound magnetization currents density.

Maxwell's equation:

$$\nabla \times \overline{H} = \overline{J} \qquad \text{(free charge)}$$

$$\nabla \times \frac{\overline{B}}{\mu_o} = \overline{J} \quad ; \quad \overline{B} = \mu_o \overline{H}$$

$$\Rightarrow \quad \nabla \times \frac{\overline{B}}{\mu_o} = \left(\overline{J} + \overline{J}_m\right)$$

due to free charges and bound magnetization currents

$$\nabla \times \overline{M} = \overline{J}_m$$

$$\therefore \quad \nabla \times \frac{\overline{B}}{\mu_o} = \left(\overline{J} + \nabla \times \overline{M}\right)$$

$$\nabla \times \left(\frac{\overline{B}}{\mu_o} - \overline{M}\right) = \overline{J}$$

Define:

$$\overline{H} \cong \left(\frac{\overline{B}}{\mu_o} - \overline{M}\right)$$

$$\therefore \quad \nabla \times \overline{H} = \overline{J}$$

Hence:

$$\overline{B} = \mu_o(\overline{H} + \overline{M})$$

Magnetization in isotropic material:

$$\overline{M} = \chi_m \overline{H}$$

χ_m = magnetic susceptibility

Hence:

$$\overline{B} = \mu_o \overline{H}(1 + \chi_m)$$

$$\mu_r \cong (1 + \chi_m)$$

$$\therefore \quad \overline{B} = \mu \overline{H}$$

$\mu = \mu_o \mu_r = permeability$

Ex. 8.4: **A slab of magnetic material is found in the region given by 0 ≤ z ≤ 2 m with μ_r = 2.5. If** $\overline{B} = 10y\hat{x} - 5x\hat{y}$ mWb/m^2 **in the slab, determine:**

$(a)\ \overline{J}\quad (b)\ \overline{J}_m\quad (c)\ \overline{M}\quad (d)\ \overline{J}_{sm}\quad \text{on } z = 0$

Solution:

$$(a)\ \overline{J} = \nabla \times \overline{H} = \nabla \times \frac{\overline{B}}{\mu_0 \mu_r} = \frac{1}{4\pi \times 10^{-7}(2.5)}\left(\frac{dB_y}{dBx} - \frac{dB_x}{dy}\right)\hat{z}$$

$$= \frac{10^6}{\pi}(-5-10)10^{-3}\hat{z} = -4.775\,\hat{z}\ \text{kA/m}^2$$

$$(b)\ \overline{J}_m = \chi_m \overline{J} = (\mu_r - 1)\overline{J} = 1.5(-4.775\hat{z})\cdot 10^3 = -7.163\hat{z}\ \text{kA/m}^2$$

$$(c)\ \overline{M} = \chi_m \overline{H} = \chi_m \frac{\overline{B}}{\mu_0 \mu_r}$$

$$= \frac{1.5(10y\hat{x} - 5x\hat{y})\cdot 10^{-3}}{4\pi \times 10^{-7}(2.5)}$$

$$= 4.775\,y\hat{x} - 2.387\,x\hat{y}\ \text{kA/m}$$

$$(c)\quad \overline{M} = 4.775\, y\hat{x} - 2.387\, x\hat{y}\ \text{kA/m}$$

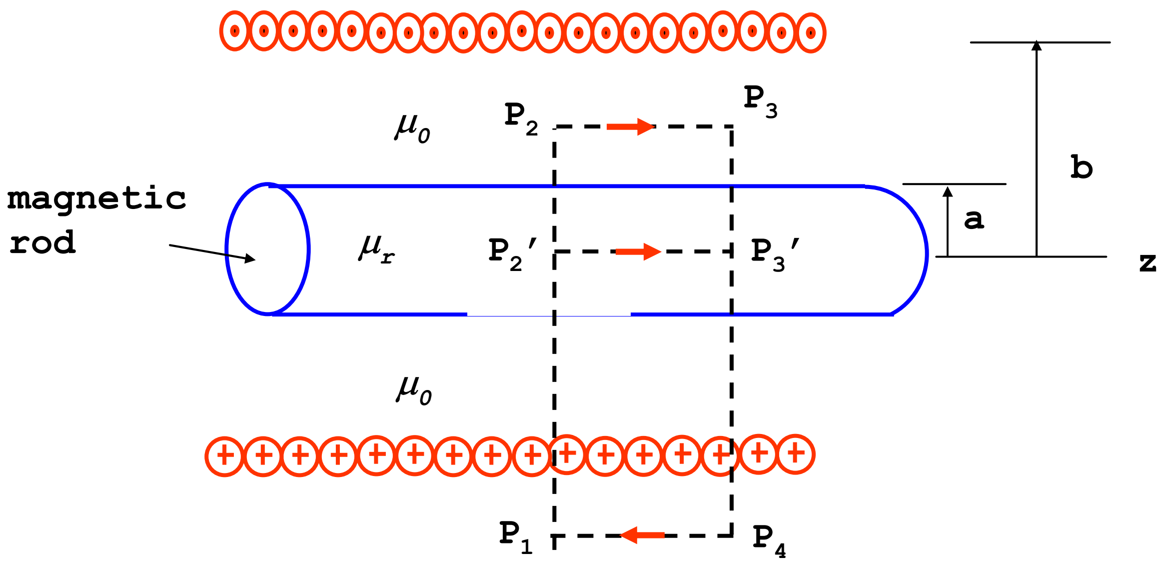
(d) $$\overline{J}_{sm} = \overline{M} \times \hat{n}$$

Because of z = 0 is under the slab region of 0 ≤ z ≤ 2 , therefore $\hat{n} = -\hat{z}$

$$\overline{J}_{sm} = = 2.387\, x\hat{x} + 4.775\, y\hat{y}\ \text{kA/m}$$

$$\left(4.775\, y\hat{x} - 2.387\, x\hat{y}\right) \times \left(-\hat{z}\right)$$

Ex. 8.5: **A closely wound long solenoid has a concentric magnetic rod inserted as shown in the diagram. In the center region, find:** (a) $\overline{H}, \overline{B}$ and $\overline{M}$ **in both air and magnetic rod, (b) the ratio of the** $\overline{B}$ **in the rod to the** $\overline{B}$ **in the air, (c)** $\overline{J}_{sm}$ **on the surface of the rod and** $\overline{J}_m$ **within the rod. Assume the permeability of the rod equals** $5\mu_o$ **.**



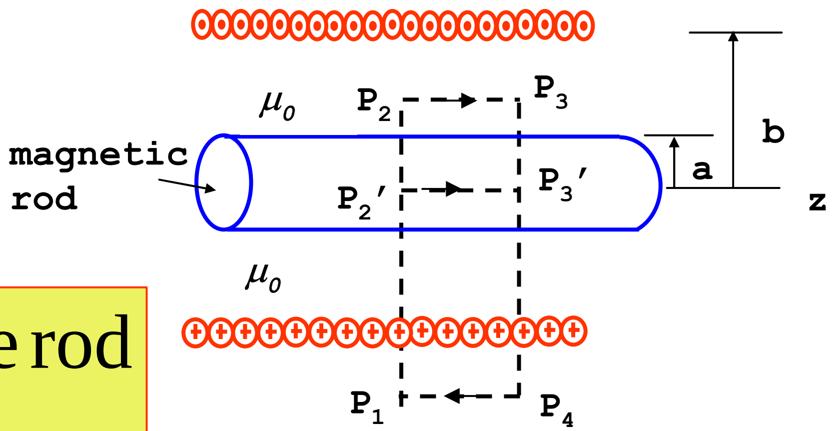
Solution:

(a) Using Ampere's circuital law to the closed path P_1 *-* P_2 *-* P_3 *-* P_4*. If using path* P_1 *-* P_2' *-* P_3' *-* P_4 *-* P_1*,* H_z *in the rod will be the same as in the air since Ampere's circuital law does not include any* I_m *in its* I_{en} *term.*

Hence:

$$\oint_{\ell} \overline{H} \cdot d\overline{\ell} = \int_{P2}^{P3} (\hat{z} H_z) \cdot (\hat{z} dz) = H_z d = I_{en} = \frac{NI}{\ell} d$$

$$H_z = \frac{NI}{\ell} = J_s$$



$$\overline{M} = \hat{z} M_z = \chi_m (\hat{z} H_z) \quad \text{in the rod}$$

$$\overline{M} = 0 \quad \text{in air}$$ ***(since χ_m of air is zero)***

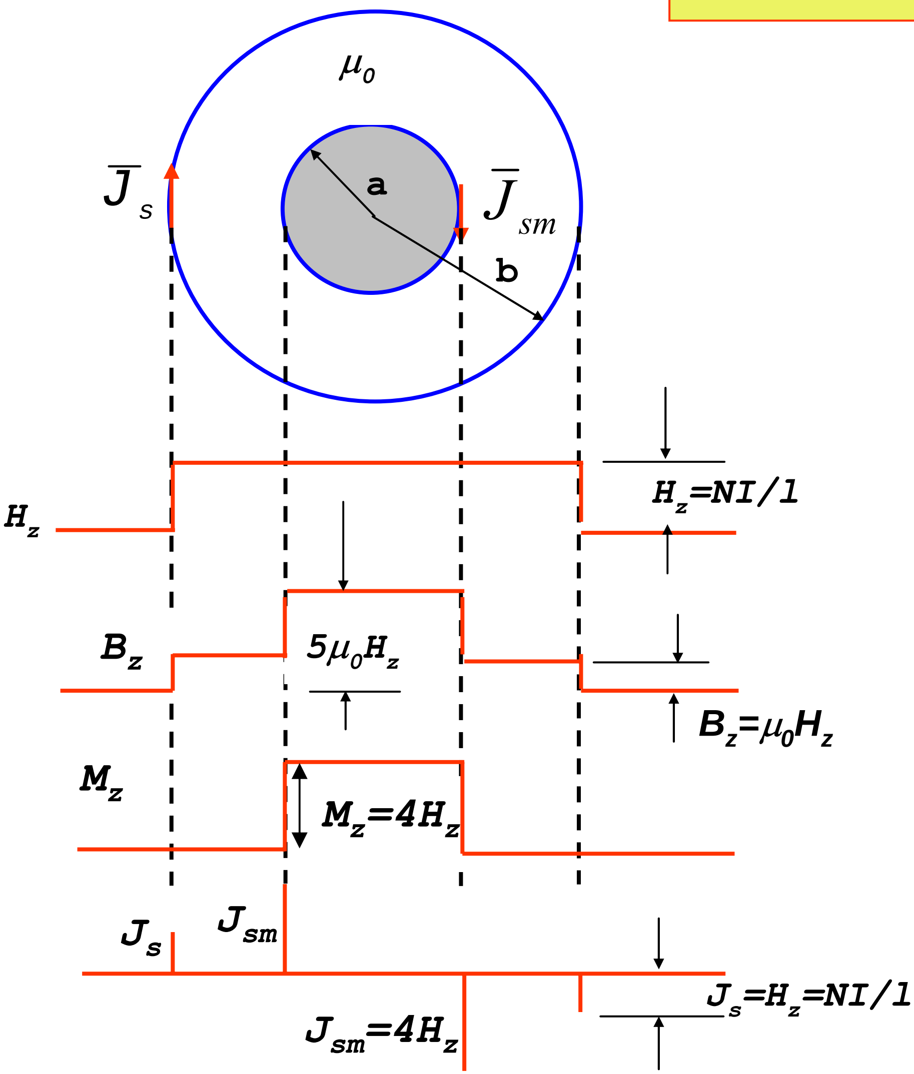
$$\overline{B} = \mu_0 (\hat{z} H_z)$$ ***in air***

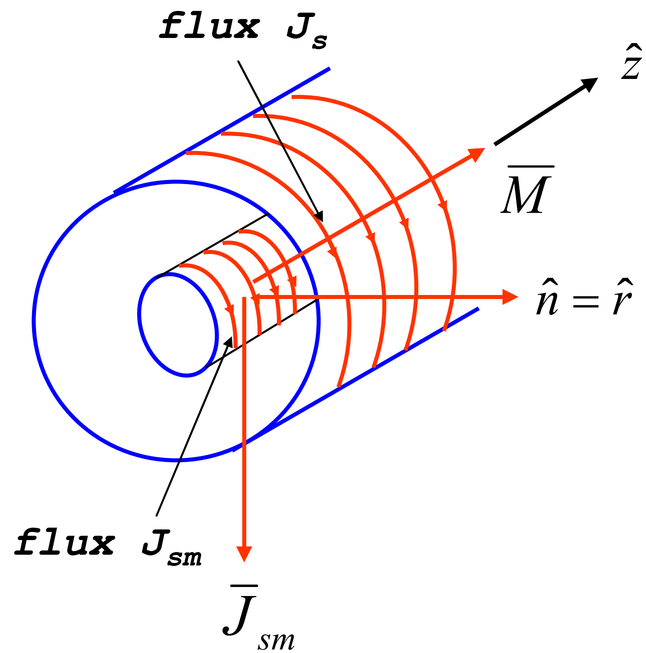
$$\overline{B} = \mu_0 \mu_r (\hat{z} H_z)$$ ***in the rod***

$$(b)\quad \frac{\bar{B}_{rod}}{\bar{B}_{air}} = \frac{5\mu_o(\hat{z}H_z)}{\mu_o(\hat{z}H_z)} = 5$$

$$(c)\quad \bar{J}_{sm} = \overline{M}\times\hat{n} = (\hat{z}M_z)\times\hat{r}_c = \hat{\phi}M_z$$

$$\bar{J}_m = \nabla\times\overline{M} = \nabla\times(\hat{z}M_z) = 0$$





Plots of H, B and M, J_s and J_{sm} along the cross section of the solenoid and the magnetic rod

8.4.5 MAGNETIC MATERIAL CLASSIFICATION

Magnetic material can be classified into two main groups:

Group A – has a zero dipole moment

$\overline{dm} = 0$ ***diamagnetic material eg. Bismuth***

$\chi_m = -1.66 \times 10^{-5}, \mu_r = 0.9999834$

Group B – has a non zero dipole moment

(a) Paramagnetic material - $\overline{dm} \neq 0$; $\overline{M} = 0$

When $\overline{B_a}$ ***is applied, there will be a slight alignment of the atomic dipole moment to produce*** $M \neq 0$

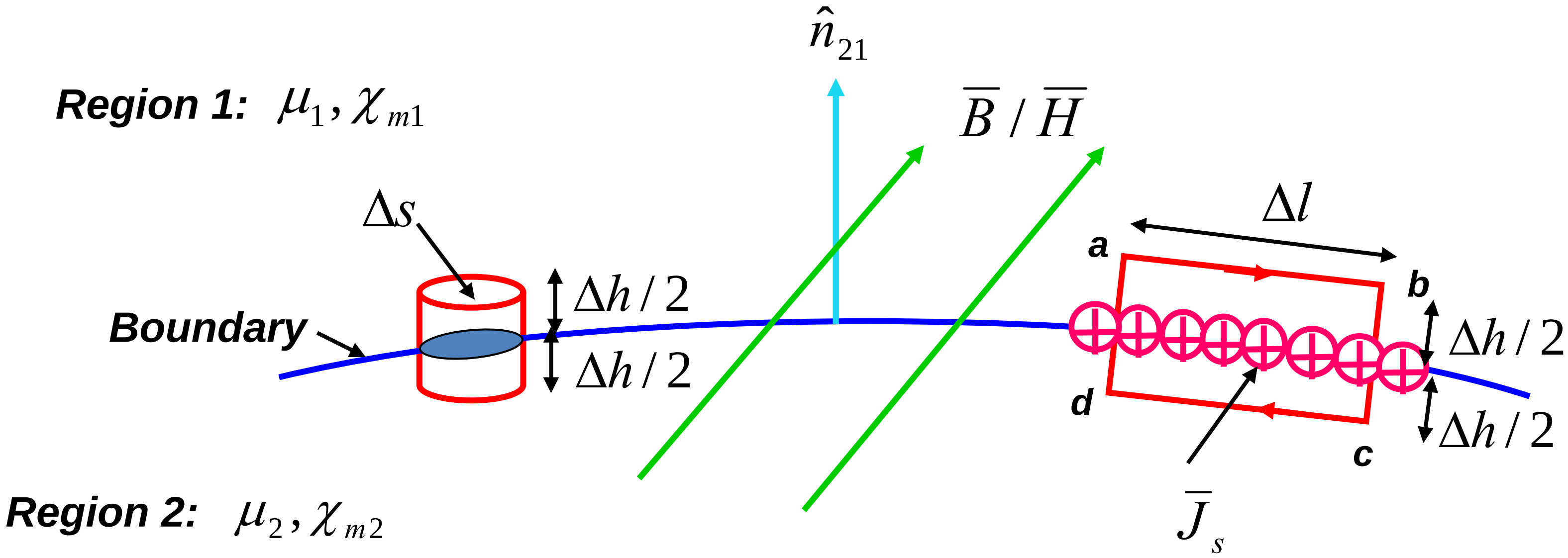
Eg. Aluminum - $\chi_m = 2 \times 10^{-5}, \mu_r = 1.00002$

(b) Ferromagnetic material : has strong magnetic moment $\overline{dm}$ ***in the absence of an applied*** B_a ***field.***

Eg: metals such as nickel, cobalt and iron.

8.5 MAGNETIC BOUNDARY CONDITIONS

To find the relationship between $\overline{B}, \overline{H}$ and $\overline{M}$



To find normal component of $\overline{B}$ and $\overline{H}$ ***at the boundary***

Consider a small cylinder as $\Delta h \to 0$ ***and use*** $\oint \overline{B} \cdot \overline{ds} = 0$

$$\oint \overline{B} \cdot \overline{ds} = B_{1n} \Delta s - B_{2n} \Delta s = 0$$

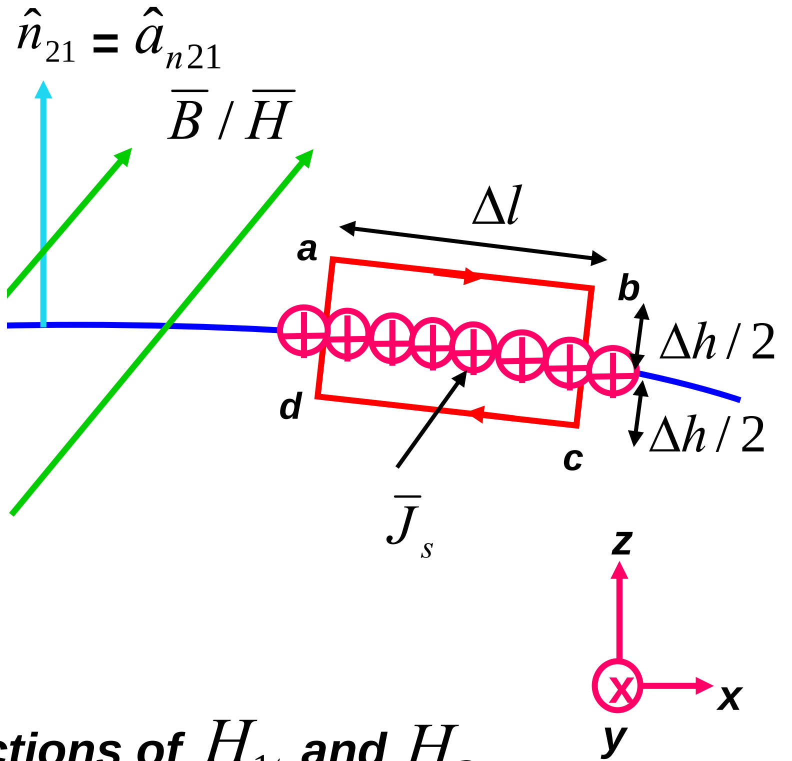
$$\therefore \quad B_{1n} = B_{2n}$$

$$\Rightarrow \mu_1 H_{1n} = \mu_2 H_{2n}$$

To find tangential component of $\overline{B}$ and $\overline{H}$ at the boundary

Consider a closed abcd as $\Delta h \rightarrow 0$

and use $\oint_l \overline{H} \cdot \overline{dl} = I_{enc}$



$$H_{1t}\Delta l - H_{2t}\Delta l = I_{enc}$$

$$\therefore \quad H_{1t} - H_{2t} = J_s$$

where J_s is perpendicular to the directions of H_{1t} and H_{2t}

$$\hat{z} \times (\hat{x}) = \hat{y}$$

In vector form :

$$\hat{a}_{n21} \times (H_1 - H_2) = \overline{J}_s$$

$\hat{a}_{n21}$ is a normal unit vector from region 2 to region 1

We have:

$$H_{1t} - H_{2t} = J_s$$

Hence:

$$\frac{B_{1t}}{\mu_1} - \frac{B_{2t}}{\mu_2} = J_s$$

We have:

$$\nabla \times \overline{M} = \overline{J}_m$$

$$\int_v \nabla \times \overline{M}\, dV = \int_v J_m dV = I_m$$

and $\oint_l \overline{M} \cdot \overline{dl} = I_m$

Hence:

$$M_{1t} - M_{2t} = J_{sm}$$

We have:

$$\overline{M} = \chi_m \overline{H}$$

Hence:

$$\frac{M_{1t}}{\chi_{m1}} - \frac{M_{2t}}{\chi_{m2}} = J_s$$

We have:

$$\mu_1 H_{1n} = \mu_2 H_{2n}$$

Hence:

$$\mu_1 \frac{M_{1n}}{\chi_{m1}} = \mu_2 \frac{M_{2n}}{\chi_{m2}}$$

If $J_s = 0$:

$$\overline{H}_{1t} = \overline{H}_{2t} \quad \text{or} \quad \frac{\overline{B}_{1t}}{\mu_1} = \frac{\overline{B}_{2t}}{\mu_2}$$

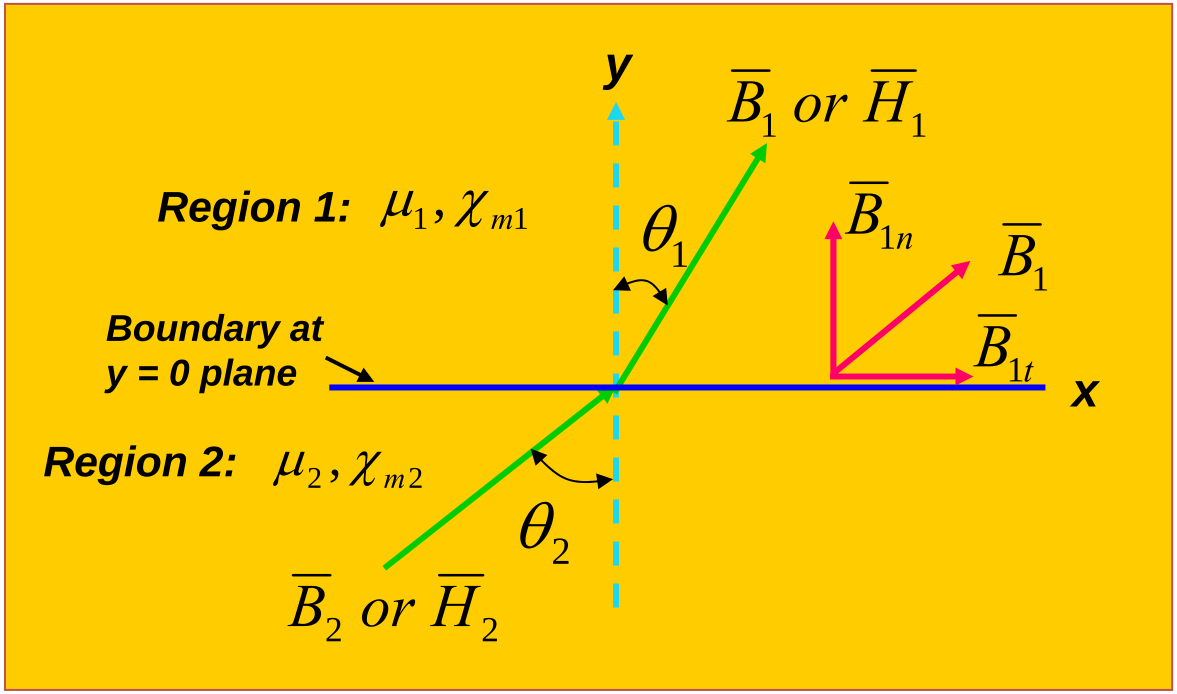
If the fields were defined by an angle θ normal to the interface

$$B_1 \cos\theta_1 = B_{1n} = B_{2n} = B_2 \cos\theta_2 \qquad \textbf{(1)}$$

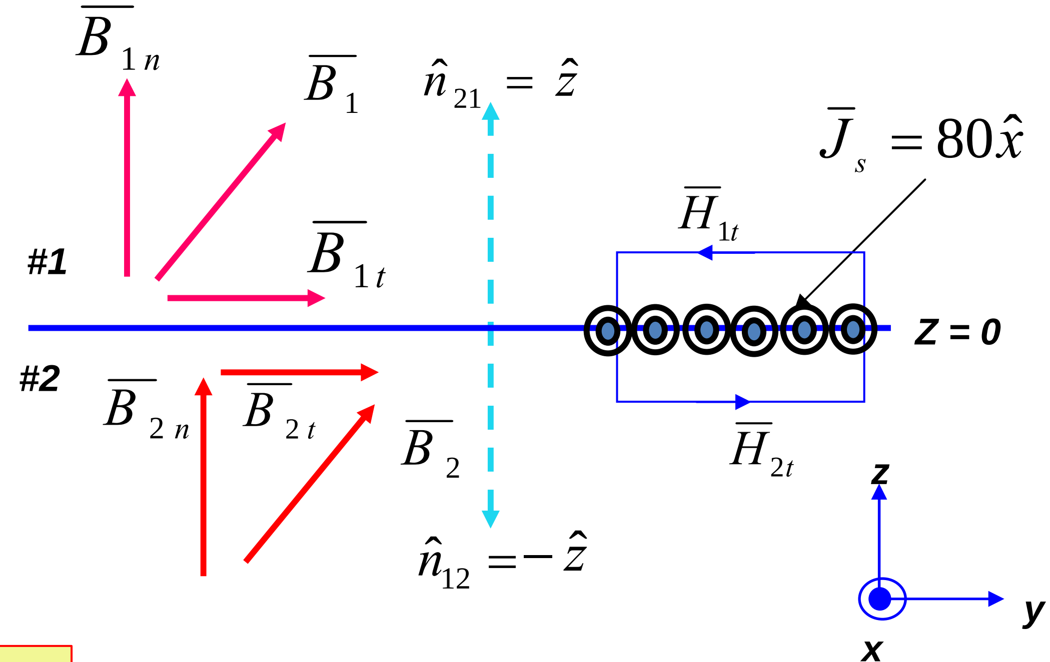
$$\frac{B_1}{\mu_1}\sin\theta_1 = H_{1t} = H_{2t} = \frac{B_2}{\mu_2}\sin\theta_2 \qquad \textbf{(2)}$$

Divide (2) to (1):

$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{\mu_1}{\mu_2} = \frac{\mu_{r1}}{\mu_{r2}}$$



Ex. 8.6: **Region 1 defined by z > 0 has μ_1 = 4 μH/m and μ_2 = 7 μH/m in region 2 defined by z < 0. $\bar{J}_s = 80\hat{x}$ A/m on the surface at z = 0. Given $\bar{B}_1 = 2\hat{x} - 3\hat{y} + \hat{z}$ mT, find $\bar{B}_2$**



Solution:

Normal component $\bar{B}_1$

$$\bar{B}_{1n} = (\bar{B}_1 \cdot \hat{n}_{12})\hat{n}_{12}$$
$$= [(2\hat{x} - 3\hat{y} + \hat{z}) \cdot (-\hat{z})](-\hat{z})$$
$$= \hat{z}$$
$$\therefore \bar{B}_{2n} = \bar{B}_{1n} = \hat{z}$$

$$\bar{B}_{1t} = \bar{B}_1 - \bar{B}_{1n} = 2\hat{x} - 3\hat{y}$$
$$\bar{H}_{1t} = \frac{\bar{B}_{1t}}{\mu_1} = \frac{(2\hat{x} - 3\hat{y})10^{-3}}{4 \times 10^{-6}}$$
$$= 500\hat{x} - 750\hat{y} \ \text{A/m}$$

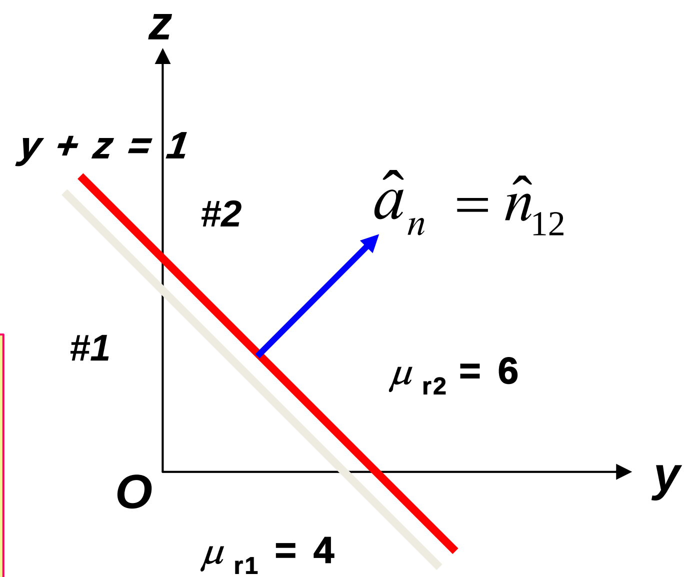
$$\bar{H}_{2t} = \bar{H}_{1t} - \hat{n}_{12} \times \bar{J}_s$$
$$= 500\hat{x} - 750\hat{y} - (-\hat{z}) \times 80\hat{x}$$
$$= 500\hat{x} - 750\hat{y} + 80\hat{y}$$
$$= 500\hat{x} - 670\hat{y} \ \text{A/m}$$

$$\bar{B}_{2t} = \mu_2 \bar{H}_{2t} = 7 \times 10^{-6}(500\hat{x} - 670\hat{y}) = 3.5\hat{x} - 4.69\hat{y}$$
$$\therefore \bar{B}_2 = \bar{B}_{2t} + \bar{B}_{2n} = 3.5\hat{x} - 4.69\hat{y} + \hat{z} \ \text{mT}$$

Ex. 8.7: **Region 1, where $\mu_{r1} = 4$ is the side of the plane $y + z < 1$. In region 2, $y + z > 1$ has $\mu_{r2} = 6$. If $\overline{B}_1 = 2\hat{x} + \hat{y}$ find $\overline{B}_2$ and $\overline{H}_2$**

Solution:

The unit normal: $\hat{n} = (\hat{y} + \hat{z}) / \sqrt{2}$



$$\overline{B}_{1n} = (\overline{B}_1 \cdot \hat{n}_{12})\hat{n}_{12}$$

$$B_{1n} = \frac{(2\hat{x} + \hat{y}) \cdot (\hat{y} + \hat{z})}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\overline{B}_{1n} = \frac{1}{\sqrt{2}}\hat{n}_{12} = 0.5\hat{y} + 0.5\hat{z} = \overline{B}_{2n}$$

$$\overline{B}_{1t} = \overline{B}_1 - \overline{B}_{1n} = 2\hat{x} + 0.5\hat{y} - 0.5\hat{z}$$

$$\overline{H}_{1t} = \frac{1}{\mu_0}(0.5\hat{x} + 0.125\hat{y} - 0.125\hat{z}) = \overline{H}_{2t}$$

$$\overline{B}_{2t} = \mu_0 \mu_{r2} \overline{H}_{2t} = 3\hat{x} + 0.75\hat{y} - 0.75\hat{z}$$

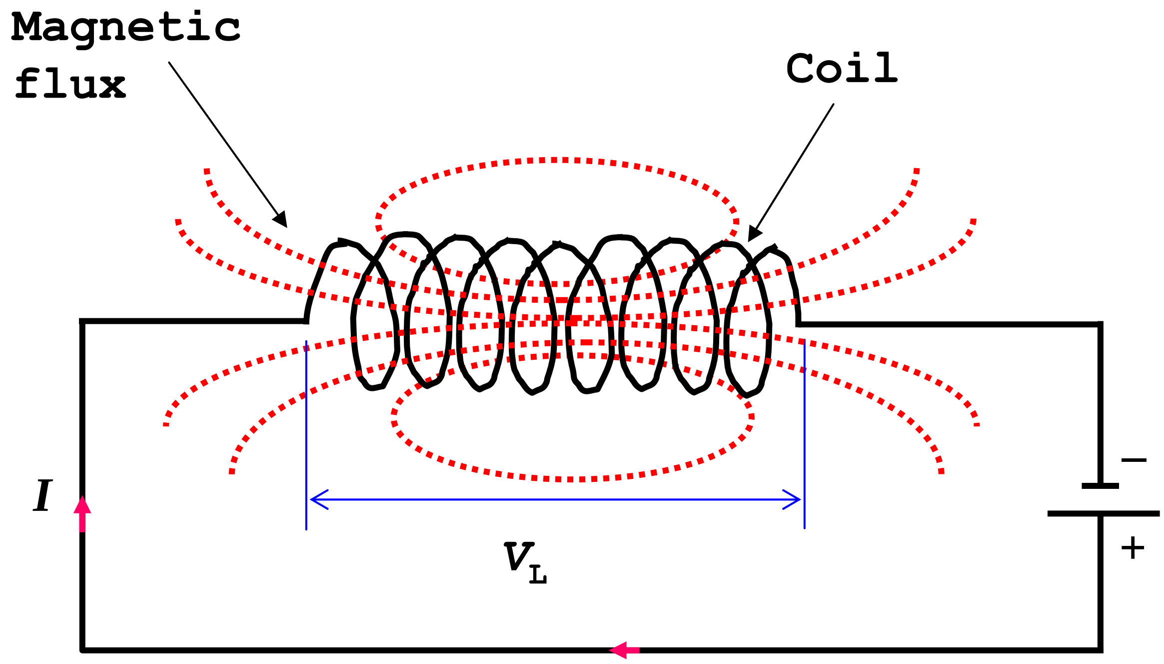
$$\therefore \overline{B}_2 = \overline{B}_{2t} + \overline{B}_{2n}$$

$$\overline{B}_2 = 3\hat{x} + 1.25\hat{y} - 0.25\hat{z} \quad \text{(T)}$$

$$\overline{H}_2 = \frac{1}{\mu_0}(0.5\hat{x} + 0.21\hat{y} - 0.04\hat{z}) \text{(A/m)}$$

8.6 SELF INDUCTANCE AND MUTUAL INDUCTANCE

Simple electric circuit that shows the effect of energy stored in a magnetic field of an inductor :



From circuit theory the induced potential across a wire wound coil such as solenoid or a toroid :

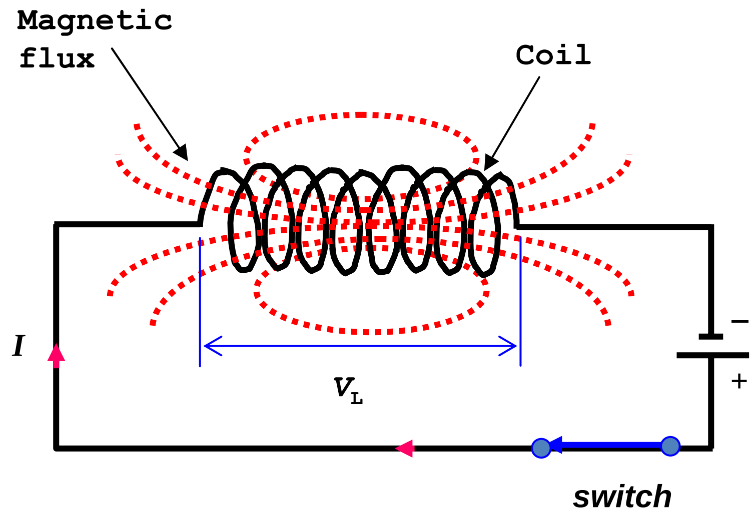
$$V_L = L\frac{dI}{dt}$$

where L is the inductance of the coil, I is the time varying current flowing through the coil – inductor.

In a capacitor, the energy is stored in the electric field :

$$W_E = \frac{1}{2}CV^2$$

In an inductor, the energy is stored in the electric field, as suggested in the diagram :



$$W_m = \int_{t=0}^{t=t_0} V_L I dt = \int_{t=0}^{t=t_0} \left(L \frac{dI}{dt} \right) I dt$$

$$= \int_{t=0}^{t=t_0} LI \, dI = \frac{1}{2} LI^2 \quad (Joule)$$

Define the inductance of an inductor :

$$L \cong \frac{\Lambda}{I} \quad Henry$$

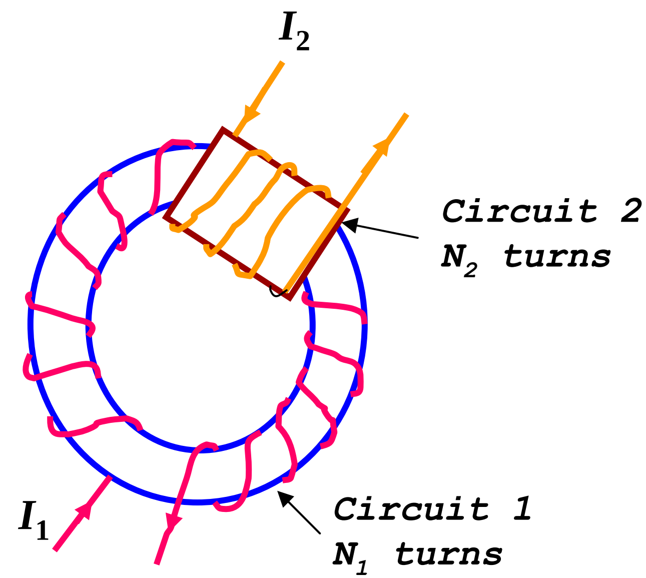
where Λ *(lambda) is the total flux linkage of the inductor*

$$L \cong \frac{\Lambda}{I} \quad H\ (Henry)$$

$$\Lambda = \psi_m N$$ **Weber turns**

Hence :

$$L = \frac{\psi_m N}{I} \quad \mathbf{H}$$



Two circuits coupled by a common magnetic flux that leads to mutual inductance.

Mutual inductance :

$$M_{12} \cong \frac{\Lambda_{12}}{I_1}$$

Λ_{12} ***is the linkage of circuit 2 produced by I_1 in circuit 1***

For linear magnetic medium
$M_{12} = M_{21}$

Ex. 8.8: ***Obtain the self inductance of the long solenoid shown in the diagram.***

Solution:

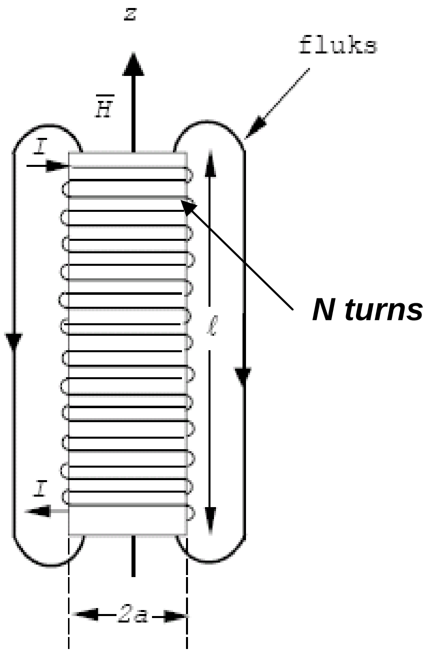
Assume all the flux ψ_m links all N turns and that $\overline{B}$ does not vary over the cross section area of the solenoid.

$$\Lambda = \psi_m N = B\left(\pi a^2\right)N$$

We have $\overline{B} = \mu\overline{H}$

$$\Lambda = (\mu H)\left(\pi a^2\right)N = \left(\frac{\mu NI}{l}\right)\left(\pi a^2\right)N$$

$$= \left(\frac{\mu N^2 I}{l}\right)\left(\pi a^2\right)$$



$$\therefore L = \frac{\Lambda}{I} = \frac{\mu N^2 \pi a^2}{l}$$

Ex. 8.9: Obtain the self inductance of the toroid shown in the diagram.

Solution:

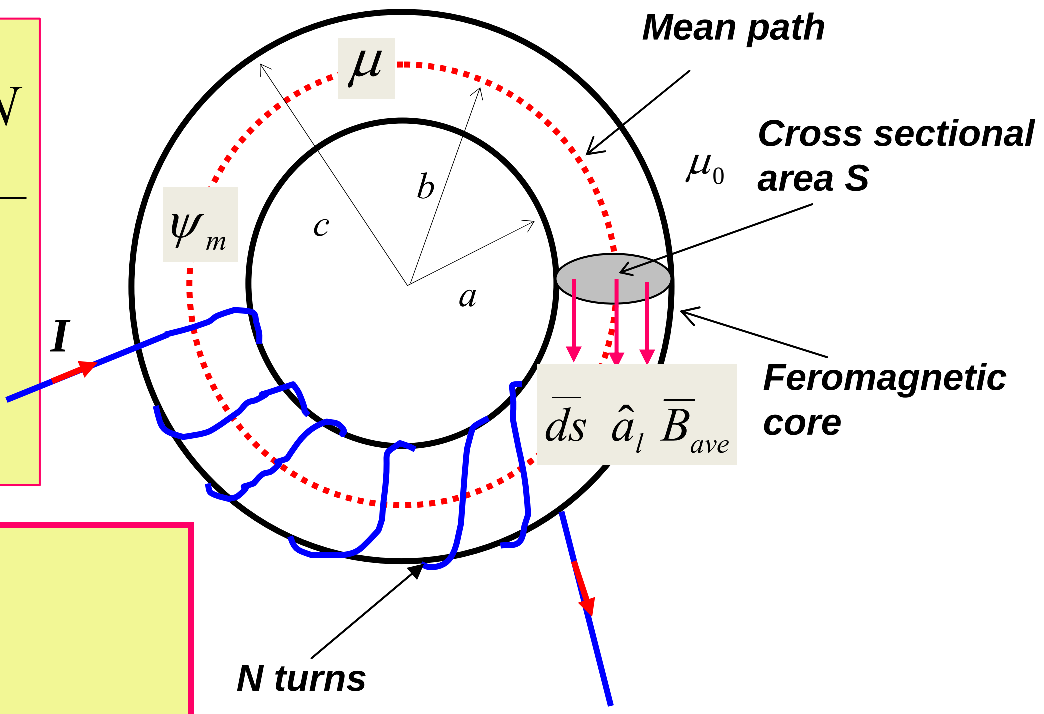
$$L = \frac{\Lambda}{I} = \frac{\psi_m N}{I} = \frac{B\left(\frac{\pi(c-a)^2}{4}\right)N}{I}$$

$$= \frac{\mu \frac{NI}{2\pi b} SN}{I}$$

$$\therefore L = \frac{\mu N^2 S}{2\pi b}$$

where b - mean radius

S - toroidal cross sectional area



Ex. 8.10: ***Obtain the expression for self inductance per meter of the coaxial cable when the current flow is restricted to the surface of the inner conductor and the inner surface of the outer conductor as shown in the diagram.***

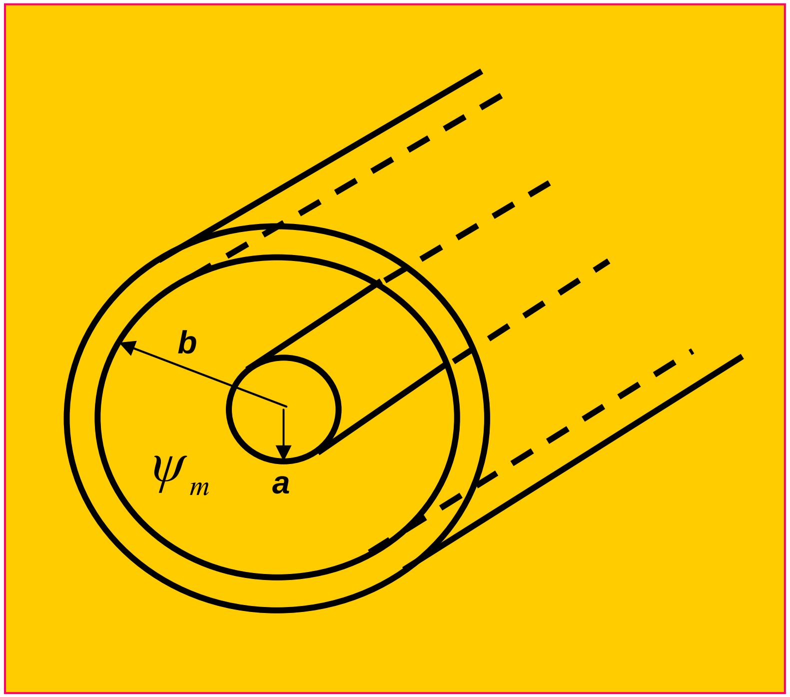
Solution:

The ψ_m will exist only between a and b and will link all the current I

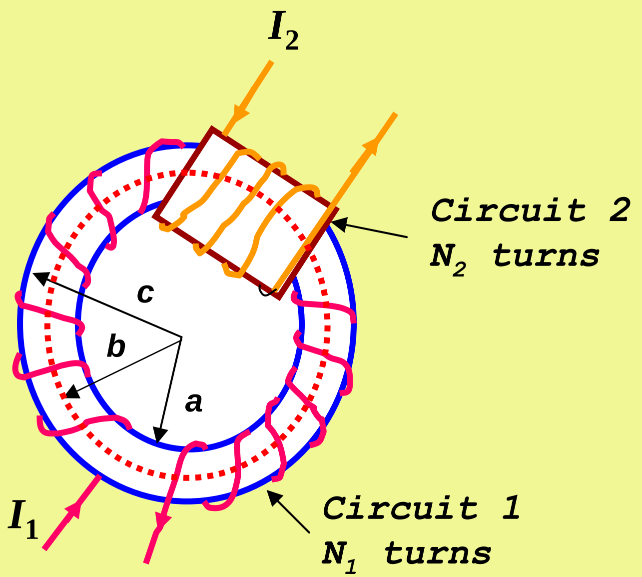
$$L = \frac{\Lambda}{I} = \frac{\psi_m}{I} = \int_0^1 \int_a^b \frac{(\mu H)(dr_c dz)}{I}$$

$$= \int_0^1 \int_a^b \frac{\mu I}{2\pi r_c} \frac{(dr_c dz)}{I}$$

$$= \frac{\mu}{2\pi} \ln \frac{b}{a}$$



Ex. 8.11: ***Find the expression for the mutual inductance between circuit 1 and circuit 2 as shown in the diagram.***



Two circuits coupled by a common magnetic flux that leads to mutual inductance.

Solution:

Let us assume the mean path :

$$2\pi b \gg (c-a)$$

$$M_{12} = \frac{\Lambda_{12}}{I_1} = \frac{\psi_{m(12)} N_2}{I_1}$$

$$= \frac{B_{12}\left(\dfrac{\pi (c-a)^2}{4}\right) N_2}{I_1}$$

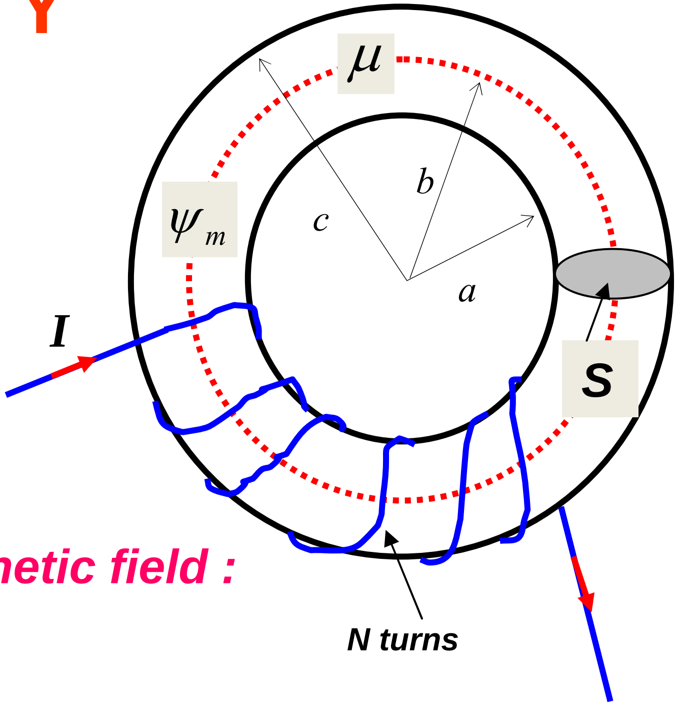
$$= \frac{\mu \dfrac{N_1 I_1}{2\pi b} S N_2}{I_1} = \frac{\mu N_1 N_2 S}{2\pi b}$$

8.7 MAGNETIC ENERGY DENSITY

We have :

$$L \cong \frac{\Lambda}{I} \quad \text{Henry}$$

$$W_m = \frac{1}{2} L I^2 = \frac{1}{2}\frac{\Lambda}{I} I^2 = \frac{1}{2}\Lambda I \quad \text{Joule}$$



Consider a toroidal ring : The energy in the magnetic field :

$$W_m = \frac{1}{2}\psi_m N I = \frac{1}{2} B S N I$$

Multiplying the numerator and denominator by 2πb :

$$W_m = \frac{1}{2} B \frac{NI}{2\pi b}\left(S 2\pi b\right)$$

where $\frac{NI}{2\pi b} = H$ and $(S2\pi b)$ is the volume V of the toroid

Hence :

$$W_m = \frac{1}{2} BHV$$

$$w_m = \frac{W_m}{V} = \frac{1}{2} BH = \frac{1}{2} \mu H^2 \qquad Jm^{-3}$$

In vector form :

$$w_m = \frac{1}{2} \overline{B} \cdot \overline{H}$$

Hence the inductance :

$$L = \frac{2}{I^2} W_m = \frac{2}{I^2} \int_v \frac{1}{2} \overline{B} \cdot \overline{H} \, dv$$

Ex. 8.12: ***Derive the expression for stored magnetic energy density in a coaxial cable with the length l and the radius of the inner conductor a and the inner radius of the outer conductor is b. The permeability of the dielectric is μ.***

Solution:

$$H = \frac{I}{2\pi r}$$

$$W_m = \frac{1}{2}\int_v \mu H^2 dv = \frac{\mu I^2}{8\pi^2}\int_v \frac{1}{r^2} dv$$

$$W_m = \frac{\mu I^2}{8\pi^2}\int_a^b \frac{1}{r^2}\cdot 2\pi r l \; dr$$

$$= \frac{\mu I^2 l}{4\pi}\ln\left(\frac{b}{a}\right) \quad \text{(J)}$$