# CHAPTER 7

# MAGNETOSTATIC FIELD (STEADY MAGNETIC)

# 7.1 INTRODUCTION - SOURCE OF MAGNETOSTATIC FIELD

Originate from:

- constant current
- permanent magnet
- electric field changing linearly with time

# Analogous between electrostatic and magnetostatic fields

| Attribute | Electrostatic | Magnetostatic |
|---|---|---|
| Source | Static charge | Steady current |
| Field | $\overline{E}$ and $\overline{D}$ | $\overline{H}$ and $\overline{B}$ |
| Factor | $\varepsilon$ | $\mu$ |
| Related Maxwell equations | $\nabla \bullet \overline{D} = \rho_v$<br>$\nabla \times \overline{E} = 0$ | $\nabla \bullet \overline{B} = 0$<br>$\nabla \times \overline{H} = \overline{J}$ |
| Potential | Scalar $V$ with $\overline{E} = -\nabla V$ | Vector $\overline{A}$ with $\overline{B} = \nabla \times \overline{A}$ |
| Energy density | $w_e = \frac{1}{2}\varepsilon E^2$ | $w_m = \frac{1}{2}\mu H^2$ |

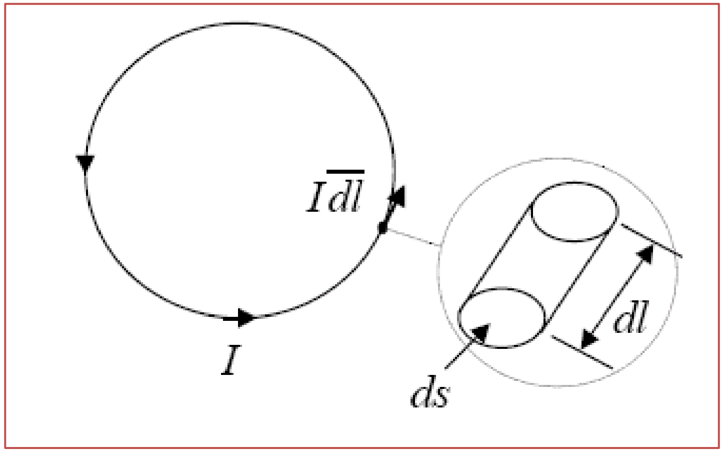
Two important laws – for solving magnetostatic field

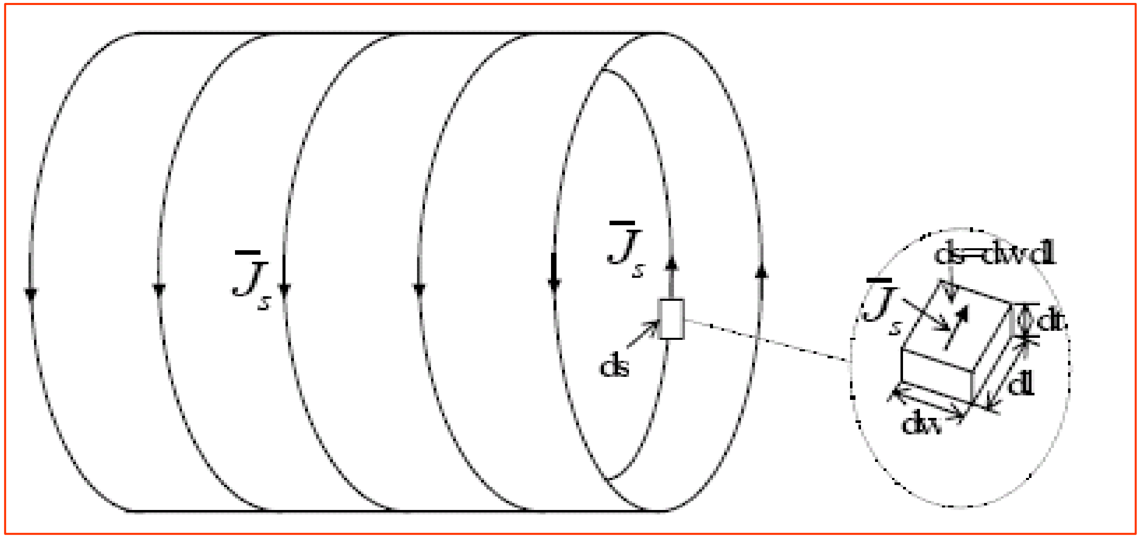
- Biot Savart Law – general case
- Ampere’s Circuital Law – cases of symmetrical current distributions

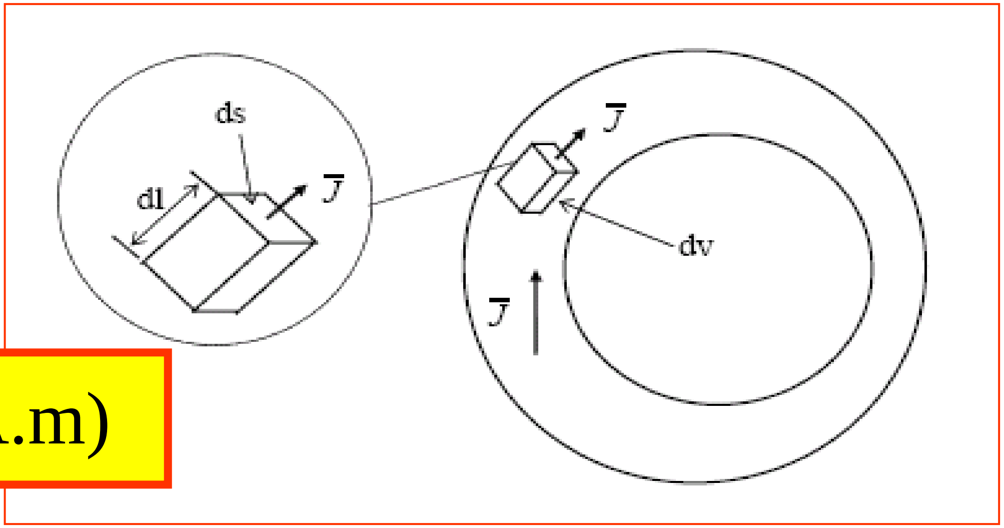
# 7.2 ELECTRIC CURRENT CONFIGURATIONS

Three basic current configurations or distributions:

- Filamentary/Line current, $I\,\overline{dl}$
- Surface current, $\overline{J}_s ds$
- Volume current, $\overline{J}dv$



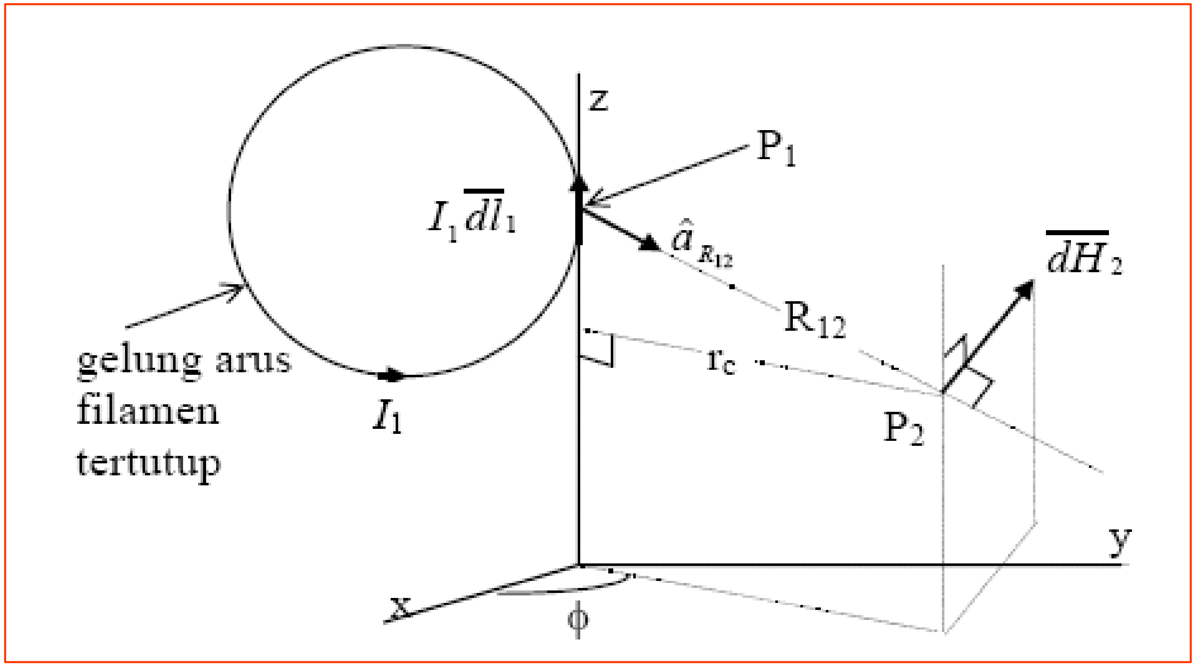




**Can be summarized:**

$$I\,\overline{dl} = \overline{J}_s ds = \overline{J}dv \quad \text{(A.m)}$$

# 7.3 BIOT SAVART LAW

**Consider the diagram as shown:**




**A differential magnetic field strength, $\overline{dH}$ results from a differential current element, $I\,\overline{dl}$. The field varies inversely with the distance squared. The direction is given by cross product of** $I\,\overline{dl}$ and $\hat{a}_R$

$$\overline{dH}_2 = \frac{I_1\,\overline{dl}_1 \text{ x } \hat{a}_{R_{12}}}{4\pi R_{12}^2} \; (\text{Am}^{-1})$$

**Total magnetic field can be obtained by integrating:**

$$\overline{H}_2 = \oint_l \frac{I\,\overline{dl} \times \hat{a}_R}{4\pi R^2} \quad (\text{Am}^{-1})$$

**Similarly for surface current and volume current elements the magnetic field intensities can be written as:**

$$\overline{dH}_2 = \frac{\overline{J}_{s_1} \times \hat{a}_{R_{12}}\, ds_1}{4\pi R_{12}^2} \quad (\text{Am}^{-1})$$

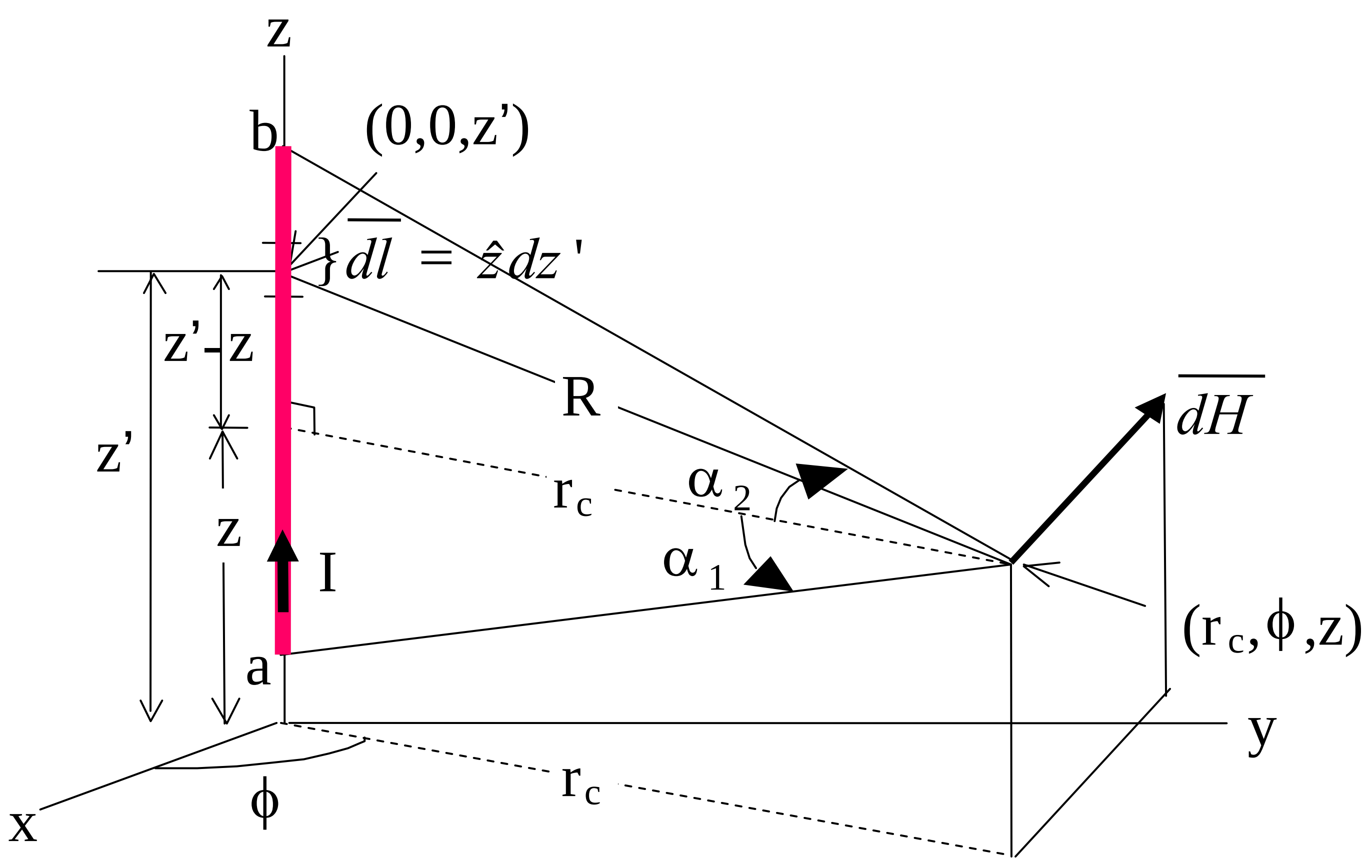
$$\Rightarrow \overline{H}_2 = \int_s \frac{\overline{J} \times \hat{a}_R\, ds}{4\pi R^2} \quad (\text{Am}^{-1})$$

$$\overline{dH}_2 = \frac{\overline{J}_1 \times \hat{a}_{R_{12}}\, dv_1}{4\pi R_{12}^2} \quad (\text{Am}^{-1})$$

$$\Rightarrow \overline{H}_2 = \int_v \frac{\overline{J} \times \hat{a}_R\, dv}{4\pi R^2} \quad (\text{Am}^{-1})$$

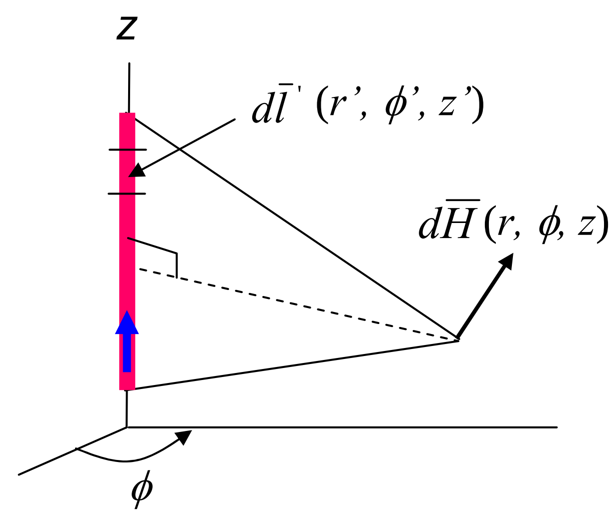
**Ex. 7.1: For a filamentary current distribution of finite length and along the z axis, find (a) $\overline{H}$ and (b) $\overline{H}$ when the current extends from -∞ to +∞.**

***Solution:***



$$\overline{H}(r_c,\phi,z) = \int_a^b \frac{I\,\overline{dl}'\ \text{x}\ \hat{a}_R(r_c',\phi',z',r_c,\phi,z)}{4\pi R^2(r_c',\phi',z',r_c,\phi,z)}$$

$$@ \qquad \overline{dH} = \frac{I(\hat{z}dz')\ \text{x}\ [\hat{r}_c r_c - \hat{z}(z'-z)]}{4\pi[r_c^{\ 2}+(z'-z)^2]^{3/2}}$$




=> $\hat{z}\ \text{x}\ \hat{r}_c = \hat{\phi}$ ; $\hat{z}\ \text{x}\ \hat{z} = 0$

$$\overline{dH} = \frac{I\hat{\phi}r_c dz'}{4\pi[r_c^{\ 2}+(z'-z)^2]^{3/2}}$$

$$\therefore \overline{H} = \frac{\hat{\phi}Ir_c}{4\pi}\int_a^b \frac{dz'}{[r_c^{\ 2}+(z'-z)^2]^{3/2}}$$

$$= \frac{\hat{\phi}I}{4\pi r_c}\left.\frac{z'-z}{[r_c^{\ 2}+(z'-z)^2]^{1/2}}\right]_a^b$$

Using Table : $\int \frac{dx}{[c^2+x^2]^{3/2}} = \frac{x}{c^2\left(c^2+x^2\right)^{1/2}}$

**Hence:**

$$\therefore \overline{H} = \frac{\hat{\phi}I}{4\pi r_c}\left\{\frac{b-z}{[r_c^{\ 2}+(b-z)^2]^{1/2}} - \frac{a-z}{[r_c^{\ 2}+(b-z)^2]^{1/2}}\right\}$$

In terms of $\alpha_1$ and $\alpha_2$ :

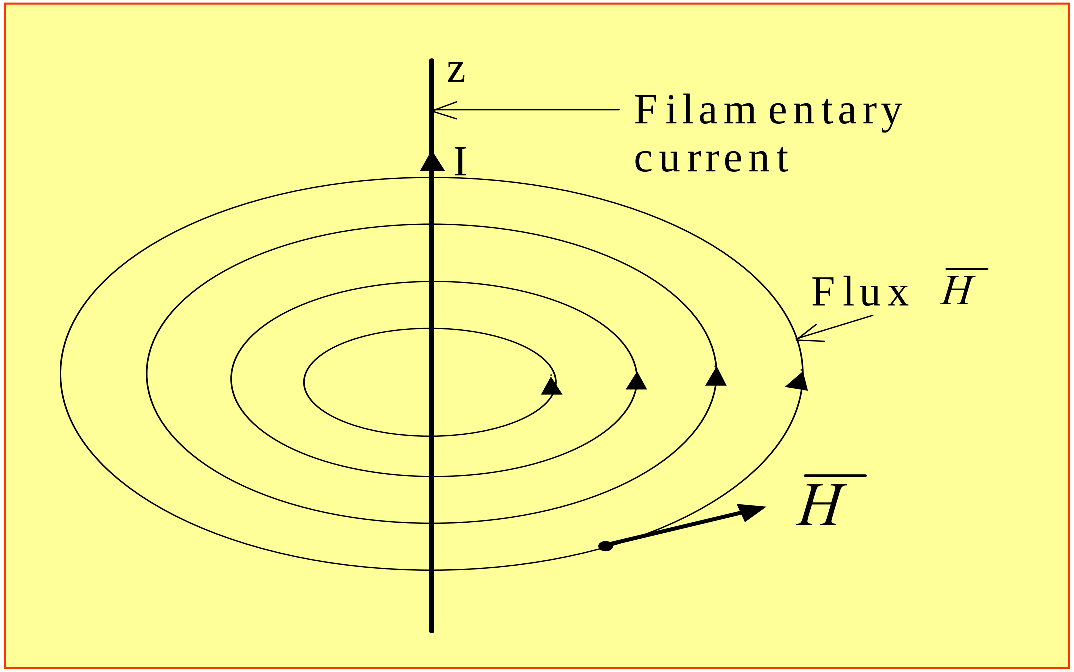
$$\overline{H} = \frac{\hat{\phi}I}{4\pi r_c}\left(\sin\alpha_2 + \sin\alpha_1\right) \quad \text{A/m}$$

(b) When a = - ∞ and b = +∞, we see that $\alpha_1 = \pi/2$, and $\alpha_2 = \pi/2$

$$\overline{H} = \frac{\hat{\phi}\ \ I}{2\pi r_c} \quad (\mathrm{Am}^{-1})$$

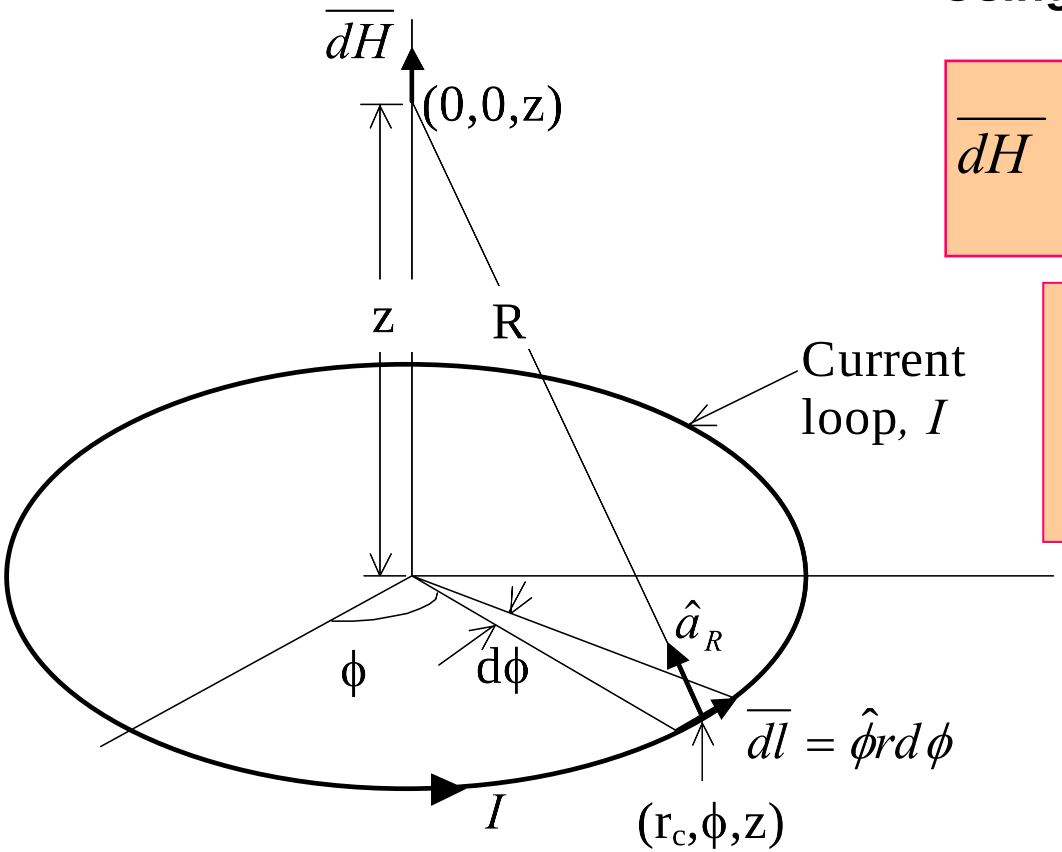
The flux of $\overline{H}$ in the $\hat{\phi}$ direction and its density decrease with $r_c$ as shown in the diagram.

$$\text{Unit vector}: \hat{\phi} = \hat{a}_l \times \hat{a}_R$$



**Ex. 7.2:** **Find the expression for the $\overline{H}$ field along the axis of the circular current loop carrying a current *I*.**

***Solution:***




**Using Biot Savart Law**

$$\overline{dH} = \frac{I(\hat{\phi} r_c d\phi)\ \mathrm{x}\ (\hat{z}z - \hat{r}\ r\ )}{4\pi[r^2 + z^2]^{3/2}}$$

**and**

$$\hat{\phi}\ \mathbf{x}\ \hat{z} = \hat{r} = \hat{x}\cos\phi + \hat{y}\sin\phi$$

$$\hat{\phi}\ \mathrm{x}\ (-\hat{r}) = \hat{z}$$

$$\therefore \overline{dH} = \hat{z}\frac{Ir^2 d\phi}{4\pi(r^2 + z^2)^{3/2}}$$

**where the $\hat{r}$ component was omitted due to symmetry**

Hence:

$$\overline{H} = \int_0^{2\pi} \overline{dH} = \frac{\hat{z} I r^2}{4\pi (r^2 + z^2)^{3/2}} \int_0^{2\pi} d\phi$$

$$= \frac{\hat{z} I r^2}{2(r^2 + z^2)^{3/2}} \qquad ; \quad r = a$$

$$= \frac{\hat{z} I a^2}{2(a^2 + z^2)^{3/2}} \quad (\text{Am}^{-1})$$

**Ex. 7.3:** **Find the $\overline{H}$ field along the axis of a s solenoid closely wound with a filamentary current carrying conductor as shown in Fig. 7.3.**

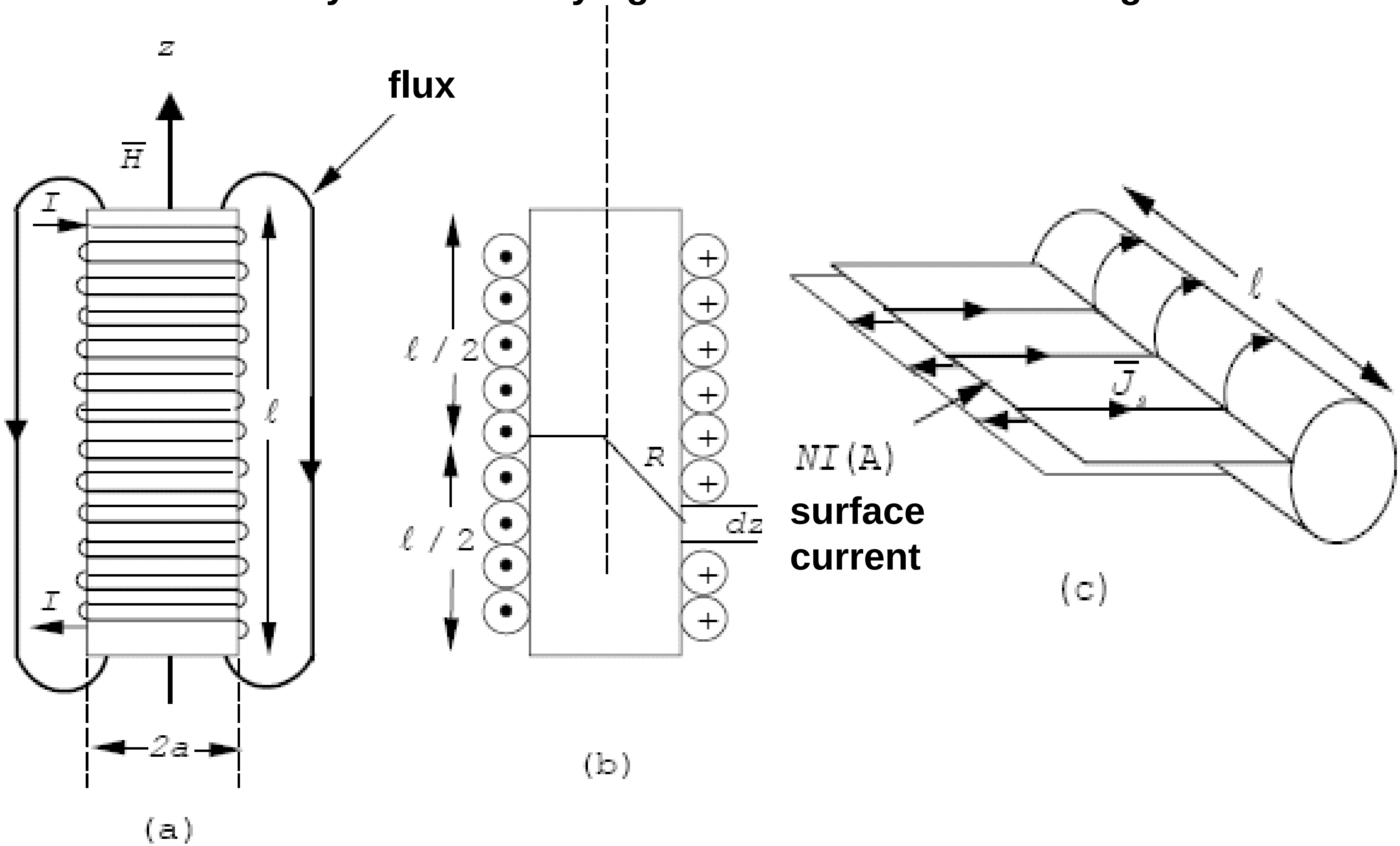
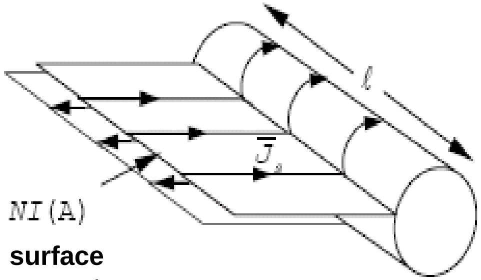



**Fig. 7.3: (a) Closely wound solenoid (b) Cross section (c) surface current, *NI* (A).**

## Solution:

- **Total surface current = *NI* Ampere**
- **Surface current density, $J_s = NI / l$ Am$^{-1}$**
- **View the *dz* length as a thin current loop that carries a current of $J_s dz = (NI / l)dz$**

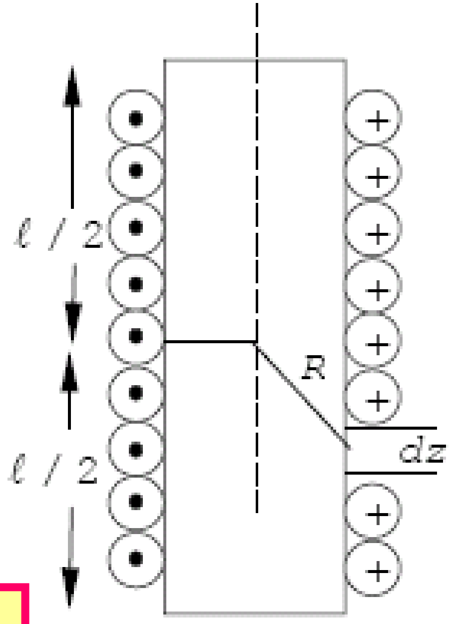



**Solution from Ex. 7.2:**

$$\overline{H} = \frac{\hat{z} I a^2}{2(a^2 + z^2)^{3/2}} \quad (\text{Am}^{-1})$$

**Therefore $d\overline{H}$ at the center of the solenoid:**

$$d\overline{H} = \hat{z} \frac{\left(\frac{NI}{\ell} dz\right) a^2}{2(a^2 + z^2)^{3/2}}$$



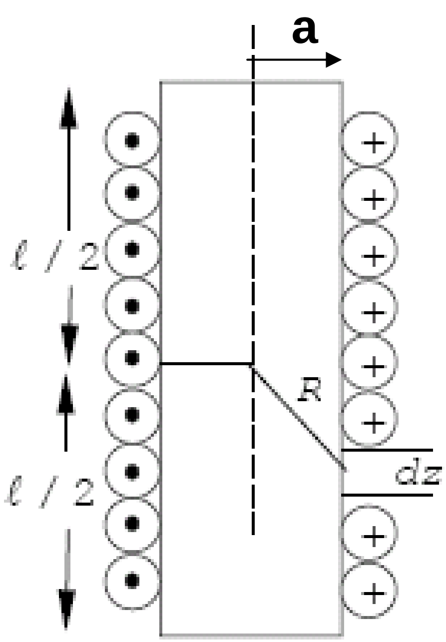

**Hence:**

$$\overline{H} = \hat{z} \frac{NIa^2}{2\ell} \int_{-\ell/2}^{\ell/2} \frac{dz}{(a^2 + z^2)^{3/2}} = \hat{z} \frac{NI}{(4a^2 + \ell^2)^{1/2}}$$

**If $\ell >> a$ :**

$$\overline{H} \approx \hat{z}\frac{NI}{\ell} = \hat{z}J_s \quad (\mathrm{Am}^{-1})$$

**at the center of the solenoid**




**Field at one end of the solenoid is obtained by integrating from 0 to $\ell$ :**

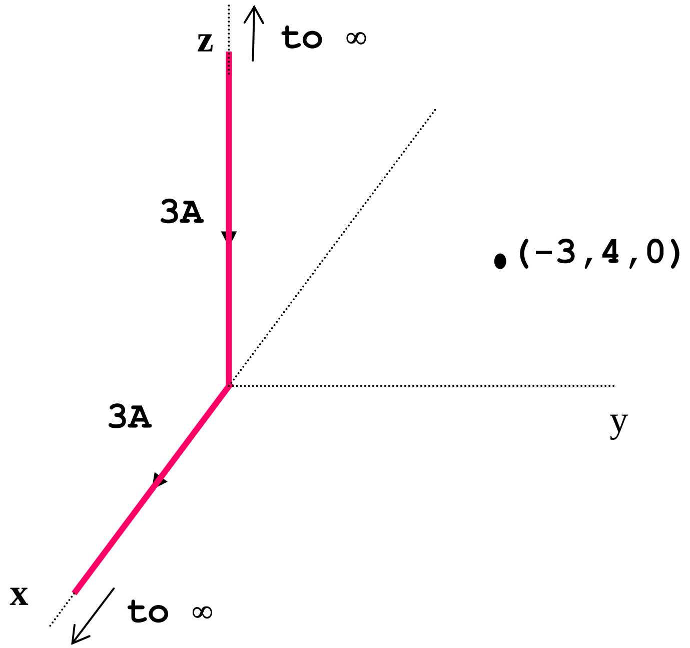
$$\overline{H} = \hat{z}\frac{NI}{2(a^2+\ell^2)^{1/2}}$$

**If $\ell >> a$ :**

$$\overline{H} \cong \hat{z}\frac{NI}{2\ell} = \hat{z}\frac{J_s}{2}$$

**which is one half the value at the center.**

**Ex. 7.4:** **Find $\overline{H}$ at point (-*3,4,0*) due to the filamentary current as shown in the Fig. below.**




***Solution:***

**Total magnetic field intensity is given by :**

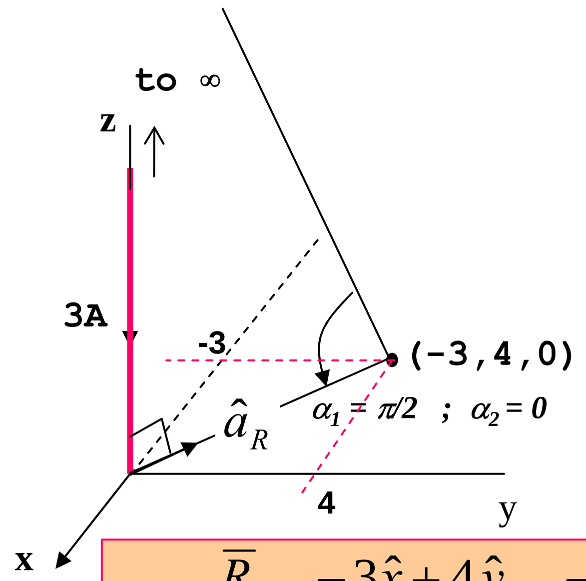
$$\overline{H} = \overline{H}_x + \overline{H}_z$$

**To find $\overline{H}_z$ :**

$$\overline{H}_z = \frac{\hat{\phi} I}{4\pi r_c}\left(\sin\alpha_2 + \sin\alpha_1\right)$$

**Unit vector:** $\hat{\phi} = \hat{a}_l \times \hat{a}_R$

$$= -\hat{z}\left(-\frac{3}{5}\hat{x} + \frac{4}{5}\hat{y}\right) = \frac{4}{5}\hat{x} + \frac{3}{5}\hat{y}$$




$$\hat{a}_R = \frac{\overline{R}}{R} = \frac{-3\hat{x} + 4\hat{y}}{\sqrt{9+16}} = \frac{-3\hat{x} + 4\hat{y}}{5}$$

**Hence:**

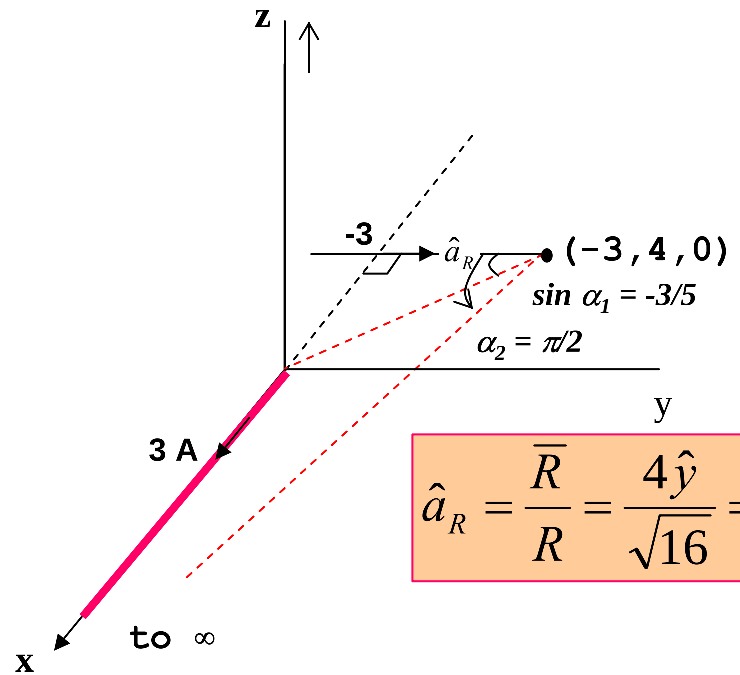
$$\overline{H}_z = \frac{\hat{\phi} I}{4\pi r_c}\left(\sin\alpha_2 + \sin\alpha_1\right)$$

$$= \left(\frac{4}{5}\hat{x} + \frac{3}{5}\hat{y}\right)\frac{3}{4\pi(5)}(0+1)$$

$$= 38.2\hat{x} + 28.65\hat{y}$$

To find $\overline{H}_x$ :

$$\overline{H}_x = \frac{\hat{\phi} I}{4\pi r_c}\left(\sin \alpha_2 + \sin \alpha_1\right)$$




**Unit vector:** $\hat{\phi} = \hat{a}_l \times \hat{a}_R$

$$= \hat{x} \times \hat{y} = \hat{z}$$

$$\hat{a}_R = \frac{\overline{R}}{R} = \frac{4\hat{y}}{\sqrt{16}} = \hat{y}$$

**Hence:**

$$\overline{H}_x = \frac{\hat{\phi} I}{4\pi r_c}\left(\sin \alpha_2 + \sin \alpha_1\right)$$

$$= \hat{z}\,\frac{3}{4\pi(4)}\left(1 - \frac{3}{5}\right)$$

$$= 23.88\,\hat{z}$$

**Hence:** $\overline{H} = \overline{H}_x + \overline{H}_z = 38.2\hat{x} + 28.65\hat{y} + 23.88\hat{z}$ A/m

# 7.4 AMPERE'S CIRCUITAL LAW

- **Solving magnetostaic problems for cases of symmetrical current distributions.**

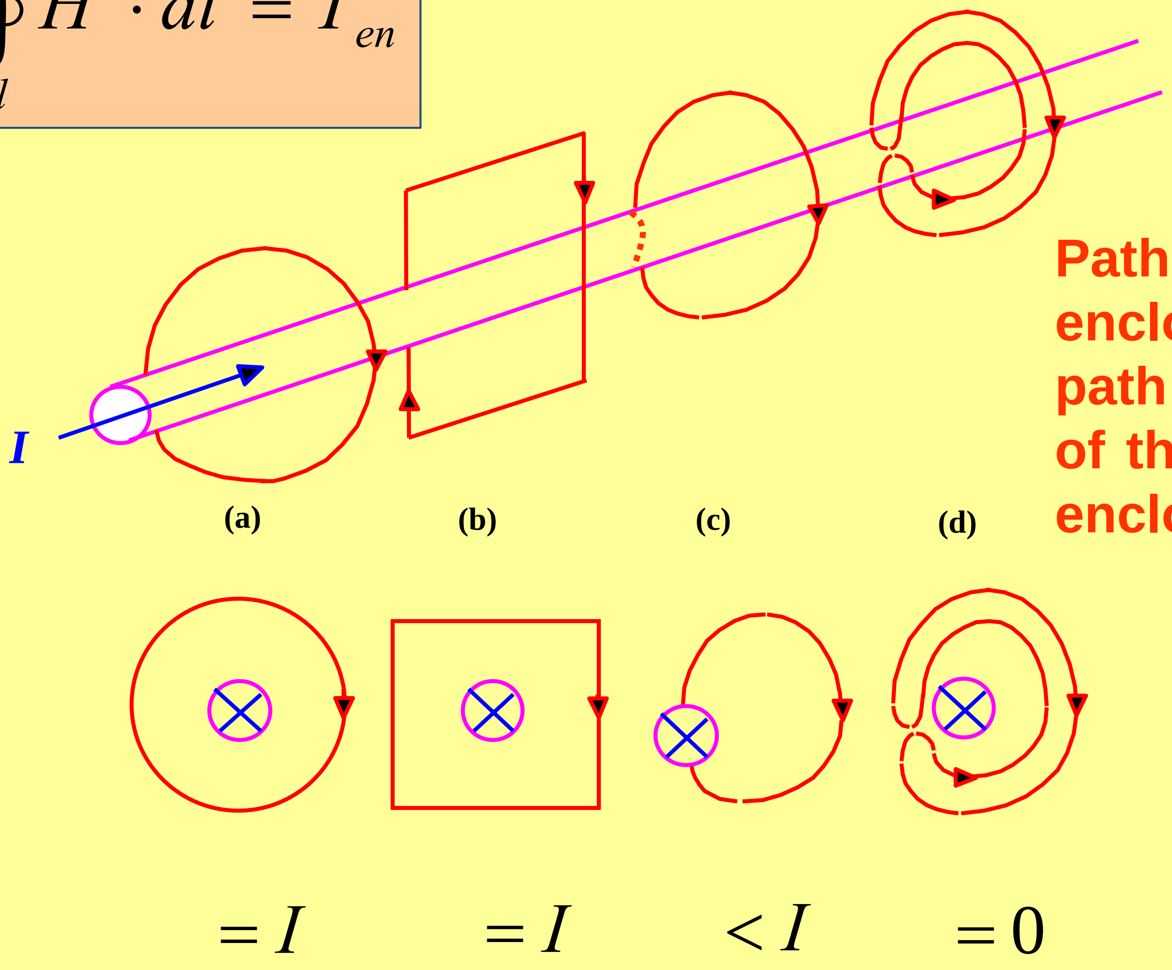
**Definition:**

***The line integral of the tangential component of the magnetic field strength around a closed path is equal to the current enclosed by the path :***

$$\oint \overline{H} \bullet \overline{dl} = I_{en}$$

# Graphical display for Ampere's Circuital Law interpretation of $I_{en}$

$$\oint_l \overline{H} \cdot \overline{dl} = I_{en}$$




Path (loop) (a) and (b) enclose the total current *I* , path c encloses only part of the current *I* and path d encloses zero current.

**Ex. 7.5:** **Using Ampere's circuital law, find $\overline{H}$ field for the filamentary current $I$ of infinite length as shown in Fig. 7.6.**

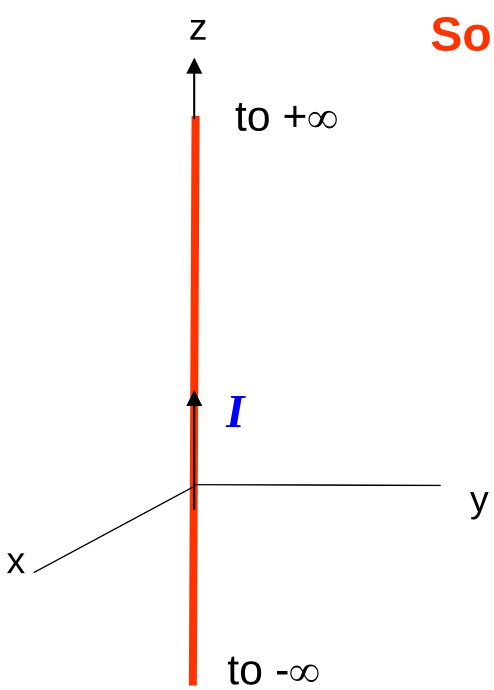



**Fig. 7.6**

**Solution:**

**Construct a closed concentric loop as shown in Fig. 7.6a.**

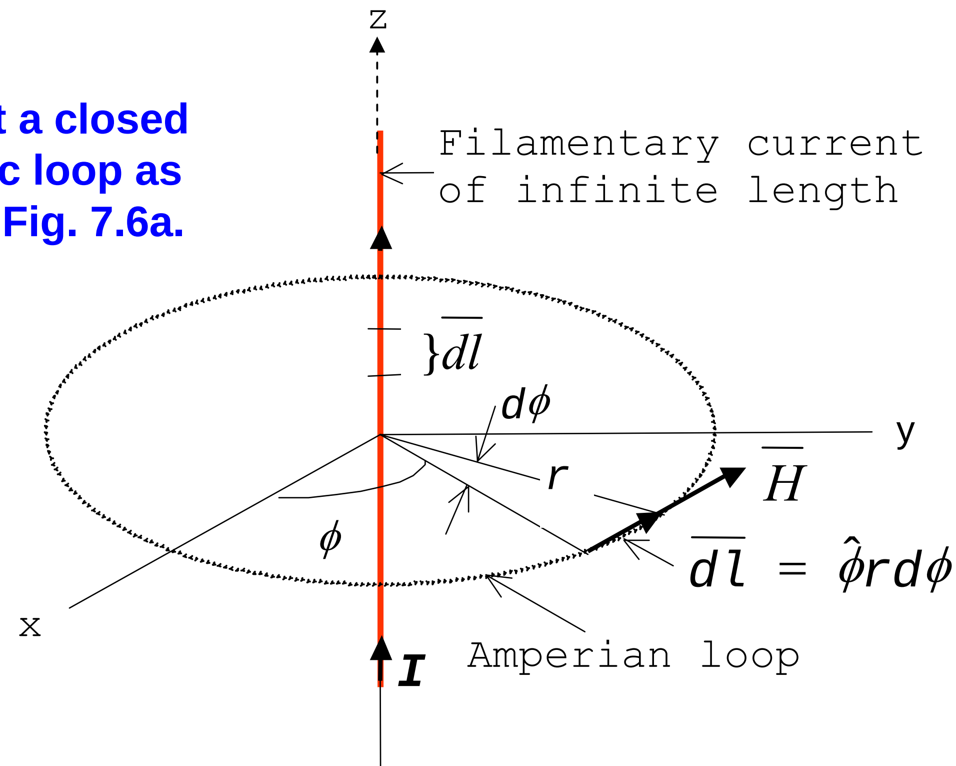



**Fig. 7.6a**

$$\oint_l \overline{H} \bullet d\overline{l} = I_{enc} = \oint_l \hat{\phi} H_\phi \bullet \hat{\phi} r d\phi = I = H_\phi r \int_0^{2\pi} d\phi = I = H_\phi (2\pi r) = I$$

$$\therefore \overline{H} = \frac{I}{2\pi r}\hat{\phi} \quad \text{(A/m)}$$ **(similar to Ex. 7.1(b) using Biot Savart)**

**Ex. 7.5:** Find $\overline{H}$ **inside and outside** an infinite length conductor of infinite cross section that carries a current ***I*** A uniformly distributed over its cross section and then **plot its magnitude**.

**Solution:**

**For *r* ≥ *a* ($C_2$) :**

$$\oint_{c_2} \overline{H} \bullet d\overline{l} = I_{enc}$$

$$H_\phi (2\pi r) = I$$

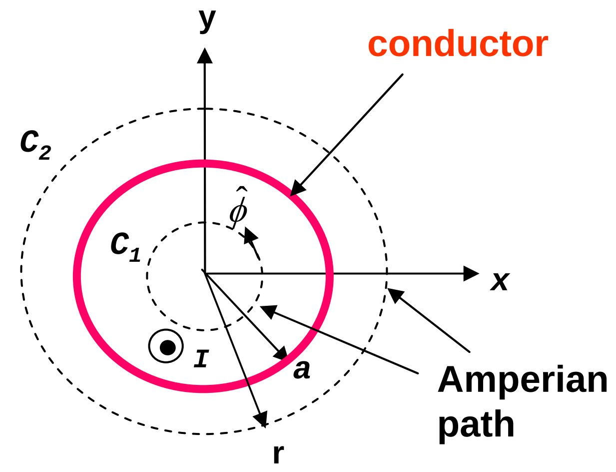
$$\therefore \overline{H} = \frac{I}{2\pi r}\hat{\phi} \quad A/m$$

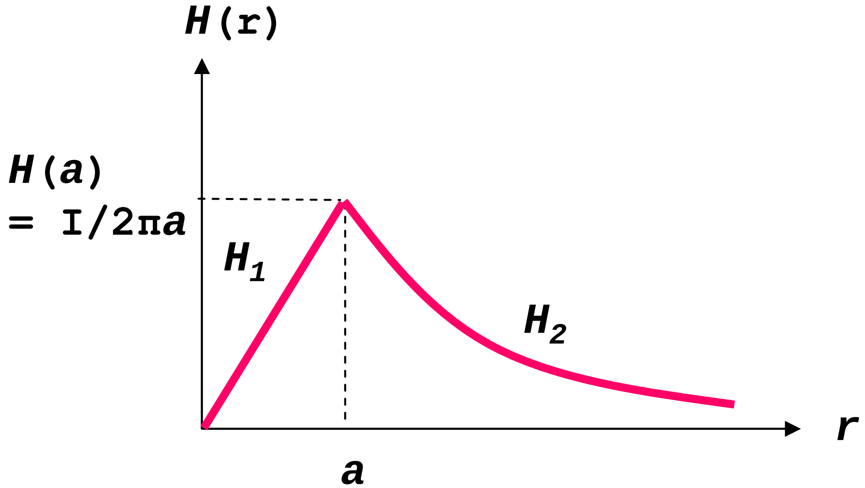
**For *r* ≤ *a* ($C_1$) :**

$$\oint_{c_1} \overline{H} \bullet d\overline{l} = I_{enc}$$

$$H_\phi (2\pi r) = I\left(\frac{\pi r^2}{\pi a^2}\right)$$

$$\therefore \overline{H} = \frac{Ir}{2\pi a^2}\hat{\phi} \quad A/m$$

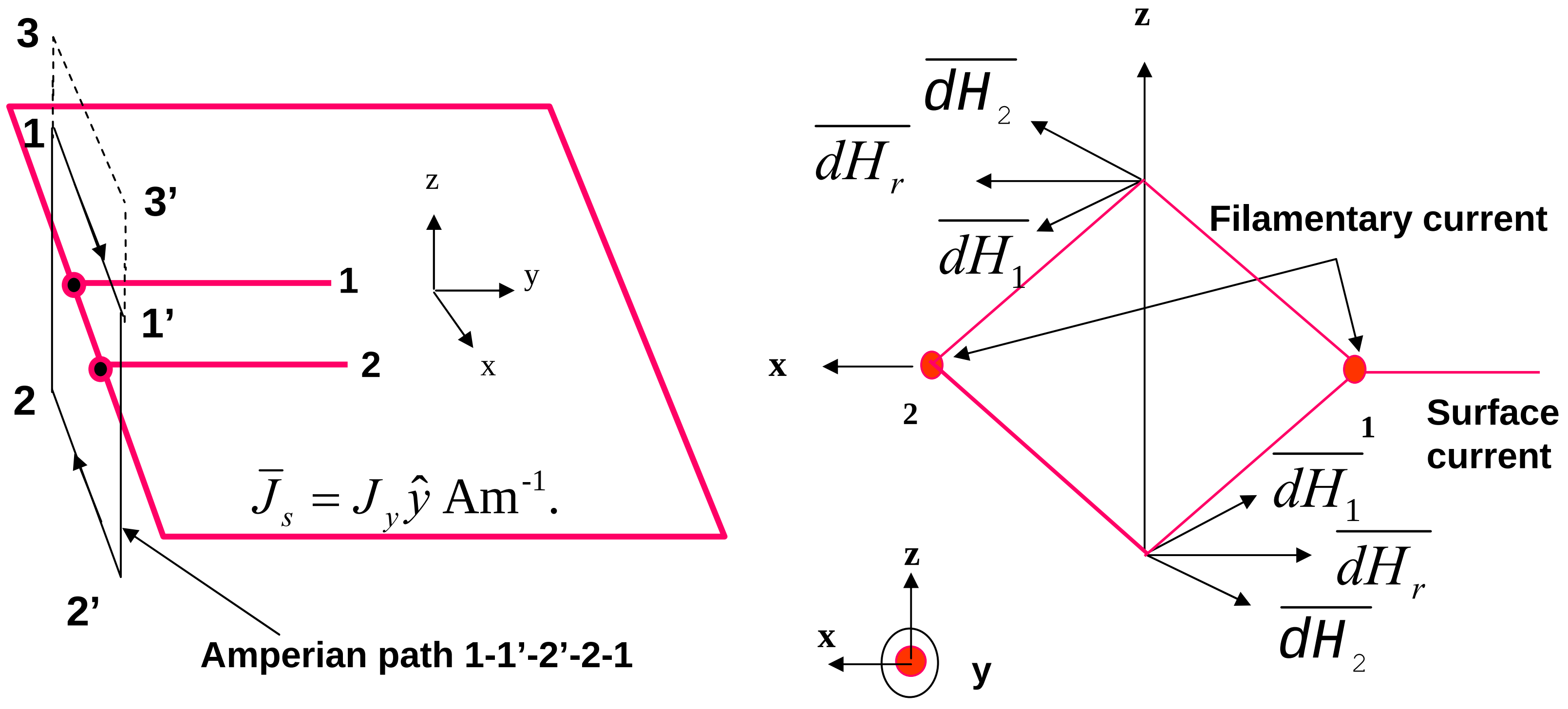


**Ex. 7.6:** Find $\overline{H}$ field above and below a surface current distribution of infinite extent with a surface current density $\overline{J}_s = J_y \hat{y}$ Am$^{-1}$.

**Solution:**

**Graphical display for finding $\overline{H}$ and using Ampere's circuital law:**




$$\oint_l \overline{H} \cdot d\overline{l} = \int_1^{1'} \overline{H} \cdot d\overline{l} + \int_{1'}^{2'} \overline{H} \cdot d\overline{l} + \int_{2'}^{2} \overline{H} \cdot d\overline{l} + \int_2^{1} \overline{H} \cdot d\overline{l} = I_{en} = J_y l$$

**From the construction, we can see that $\overline{H}$ above and below the surface current will be in the $\hat{x}$ and $-\hat{x}$ directions, respectively.**

$$\int_{1}^{1'} H_{x1}\hat{x}\cdot\hat{x}dx + \int_{2'}^{2} \overline{H}_{x2}(-\hat{x})\cdot\hat{x}dx = J_y l$$

**where**

$$\int_{1'}^{2'} \text{and} \int_{2}^{1} = 0$$

**since $\overline{H}$ is perpendicular to $\overline{dl}$**

**Therefore:**

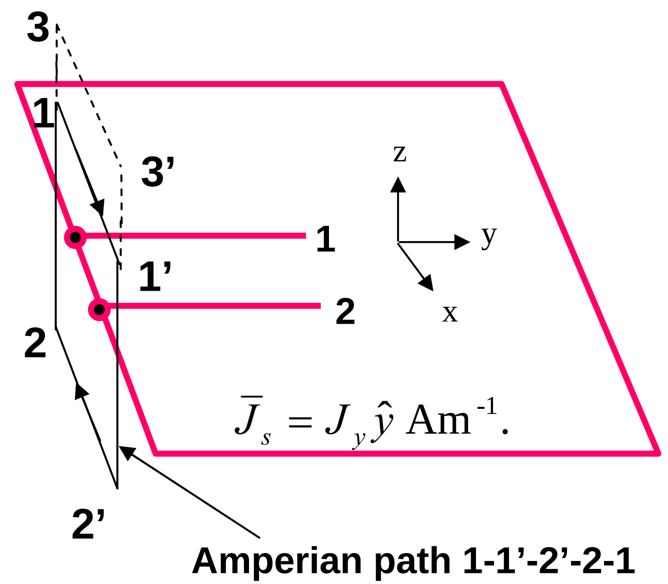
$$H_{x1}l - H_{x2}l = J_y l$$

**Similarly if we takes on the path *3-3'-2'-2-3*, the equation becomes:**

$$H_{x3}l - H_{x2}l = J_y l$$

**Hence:**

$$H_{x1} = H_{x3} = H_x$$



And we deduce that $\left|\overline{H}\right|$ above and below the surface current are equal, its becomes:

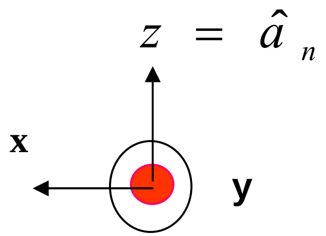
$$H_x l + H_x l = J_y l$$

$$H_x = \frac{1}{2} J_y \quad ; \quad z > 0 \qquad \overline{H} = +\frac{1}{2} J_y \, \hat{x}$$

$$H_x = -\frac{1}{2} J_y \quad ; \quad z < 0 \qquad \overline{H} = +\frac{1}{2} J_y \, (-\hat{x})$$

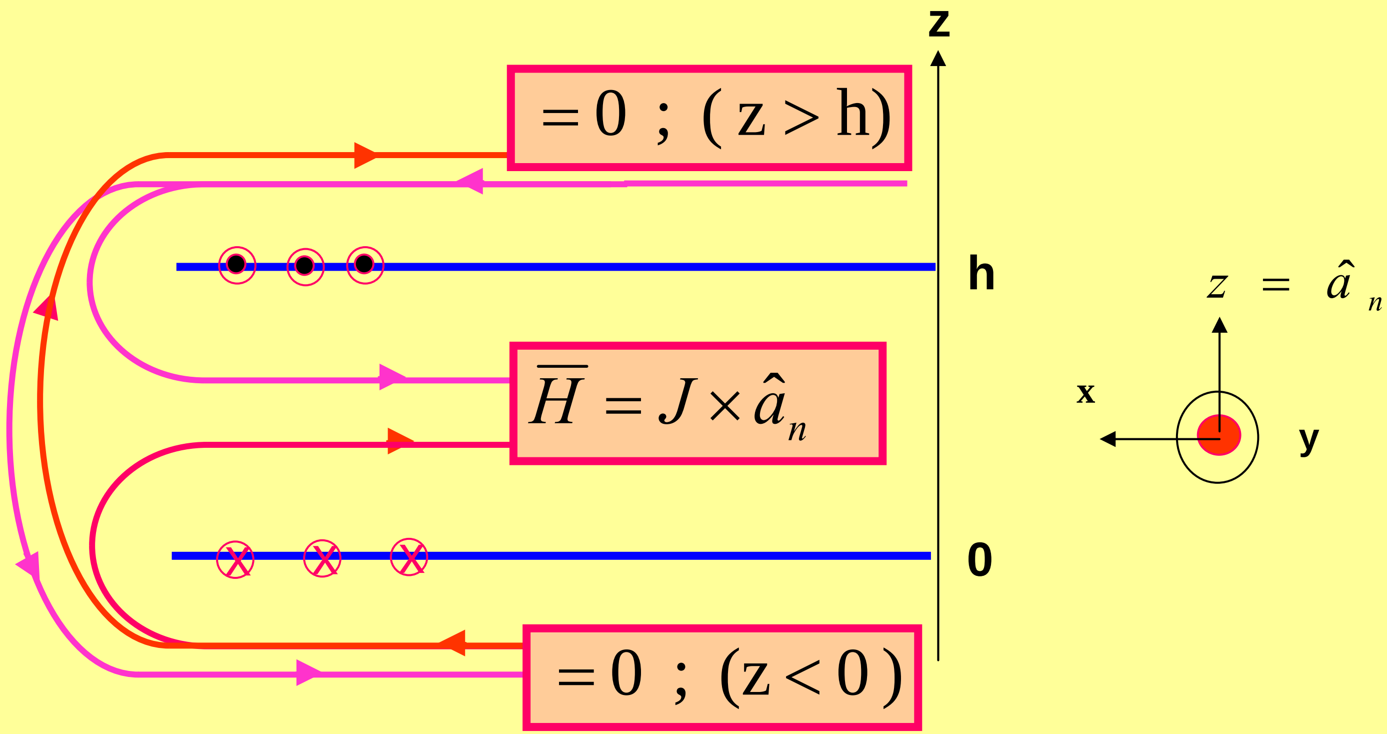
In vector form:

$$\overline{H} = \frac{1}{2} J \times \hat{a}_n$$




It can be shown for two parallel plate with separation h, carrying equal current density flowing in opposite direction the $\overline{H}$ field is given by:

$$\overline{H} = J \times \hat{a}_n \quad ; \; (0 < z < h)$$
$$= 0 \quad ; \quad (z > h \text{ and } z < 0)$$




$$\begin{aligned} \bar{H} &= J \times \hat{a}_n \quad ; \; (0 < z < h) \\ &= 0 \quad ; \quad (z > h \text{ and } z < 0) \end{aligned}$$

# 7.5 CURL (IKAL)

**The curl of a vector field, $\overline{H}$ is another vector field.**

**For example in Cartesian coordinate, combining the three components, curl $\overline{H}$ can be written as:**

$$\nabla \times \overline{H} = \left\{ \hat{x}\left[\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right] + \hat{y}\left[\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right] + \hat{z}\left[\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right]\right\}$$

**And can be simplified as:**

$$\nabla \times \overline{H} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix}$$

**Expression for curl in cyclindrical and spherical coordinates:**

$$\nabla\times\overline{H}=\left[\frac{1}{r}\left(\frac{\partial H_z}{\partial\phi}-\frac{\partial H_\phi}{\partial z}\right)\hat{r}+\left(\frac{\partial H_r}{\partial z}-\frac{\partial H_z}{\partial r}\right)\hat{\phi}+\frac{1}{r}\left(\frac{\partial}{\partial r}\left(rH_\phi\right)-\frac{\partial H_r}{\partial\phi}\right)\hat{z}\right]$$ **cyclindrical**

$$\nabla\times\overline{H}=\frac{1}{r\sin\theta}\left(\frac{\partial\left(H_\phi\sin\theta\right)}{\partial\theta}-\frac{\partial H_\theta}{\partial\phi}\right)\hat{r}$$

$$+\frac{1}{r}\left(\frac{1}{\sin\theta}\left(\frac{\partial H_r}{\partial\phi}\right)-\frac{\partial\left(rH_\phi\right)}{\partial r}\right)\hat{\theta}+\frac{1}{r}\left(\frac{\partial\left(rH_\theta\right)}{\partial r}-\frac{\partial H_r}{\partial\theta}\right)\hat{\phi}$$ **spherical**

## 7.5.1 RELATIONSHIP OF $\overline{H}$ AND $\overline{J}$

$$\nabla \times \overline{H} = \overline{J}$$

**Meaning that if** $\overline{H}$ **is known throughout a region, then** $\nabla \times \overline{H} = \overline{J}$ **will produce** $\overline{J}$ **for that region.**

**Ex. 7.7:** Find $\nabla \times \overline{H}$ for given $\overline{H}$ field as the following.

(a) $\overline{H} = \hat{\phi}\left(\dfrac{I}{2\pi r_c}\right)$ for a filamentary current

(b) $\overline{H} = \hat{\phi}\left(\dfrac{I r_c}{2\pi a^2}\right)$ in an infinite current carrying conductor with radius a meter

(c) $\overline{H} = \hat{x}\dfrac{J_s}{2}$ for infinite sheet of uniformly surface current $J_s$

(d) $\overline{H} = \hat{\phi}\dfrac{I}{2\pi r_c}\left(\dfrac{c^2 - r_c^2}{c^2 - b^2}\right)$ in outer conductor of coaxial cable

*Solution:*

**(a)** $\overline{H} = \hat{\phi}\left(\frac{I}{2\pi r_c}\right)$ **=>** $\overline{H} = \hat{\phi}H_{\phi} = \hat{\phi}\left(\frac{I}{2\pi r_c}\right),\ H_{r_c} = H_z = 0$

**Cyclindrical coordinate**

$$\nabla\times\overline{H} = \left[\frac{1}{r}\left(\frac{\partial H_z}{\partial\phi} - \frac{\partial H_{\phi}}{\partial z}\right)\hat{r} + \left(\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r}\right)\hat{\phi} + \frac{1}{r}\left(\frac{\partial}{\partial r}\left(rH_{\phi}\right) - \frac{\partial H_r}{\partial\phi}\right)\hat{z}\right]$$

**= 0** **= 0** **= 0**

**Hence:**

$$\nabla\ \mathrm{x}\ \overline{H} = \frac{\hat{z}}{r_c}\left[\frac{\partial}{\partial r_c}\left(r_c H_{\phi}\right)\right] = \frac{\hat{z}}{r_c}\left[\frac{\partial}{\partial r_c}\left(\frac{I}{2\pi}\right)\right] = 0$$

**Solution:**

**(b)** $\overline{H} = \hat{\phi}\left(\frac{Ir_c}{2\pi a^2}\right)$

**Cyclindrical coordinate**

$$\nabla \times \overline{H} = \left[\frac{1}{r}\left(\frac{\partial H_z}{\partial \phi} - \frac{\partial H_\phi}{\partial z}\right)\hat{r} + \left(\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r}\right)\hat{\phi} + \frac{1}{r}\left(\frac{\partial}{\partial r}(rH_\phi) - \frac{\partial H_r}{\partial \phi}\right)\hat{z}\right]$$

= 0 = 0 = 0

**Hence:**

$$\nabla \times \overline{H} = \frac{\hat{z}}{r_c}\left[\frac{\partial}{\partial r_c}\left(r_c \frac{Ir_c}{2\pi a^2}\right)\right] = \frac{\hat{z}}{r_c}\left(\frac{2r_c I}{2\pi a^2}\right)$$

$$= \hat{z}\frac{I}{\pi a^2} = \hat{z}J \ (\mathrm{Am}^{-2})$$

*Solution:*

**(c)** $\overline{H} = \hat{x}\frac{J_s}{2}$

**Cartesian coordinate**

$$\nabla \times \overline{H} = \left\{ \hat{x}\left[ \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \right] + \hat{y}\left[ \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \right] + \hat{z}\left[ \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right] \right\}$$

= 0 = 0 = 0

**because $H_x$ = constant and $H_y = H_z = 0$.**

**Hence:**

$$\nabla \text{ x } \overline{H} = 0$$

***Solution:***

**(d)** $$\overline{H} = \hat{\phi}\frac{I}{2\pi r_c}\left(\frac{c^2 - r_c^2}{c^2 - b^2}\right)$$

**Cyclindrical coordinate**

$$\nabla \times \overline{H} = \left[\frac{1}{r}\left(\frac{\partial H_z}{\partial \phi} - \frac{\partial H_\phi}{\partial z}\right)\hat{r} + \left(\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r}\right)\hat{\phi} + \frac{1}{r}\left(\frac{\partial}{\partial r}(rH_\phi) - \frac{\partial H_r}{\partial \phi}\right)\hat{z}\right]$$

= 0   = 0   = 0

**Hence:**

$$\nabla \times \overline{H} = \frac{\hat{z}}{r_c}\left[\frac{\partial}{\partial r_c}(r_c)\frac{I}{2\pi r_c}\left(\frac{c^2 - r_c^2}{c^2 - b^2}\right)\right]$$

$$= \frac{\hat{z}}{r_c}\left[\frac{I}{2\pi}\left(-\frac{2r_c}{c^2 - b^2}\right)\right]$$

$$= -\hat{z}\frac{I}{\pi(c^2 - b^2)} = \overline{J} \ \ (\mathrm{Am}^{-2})$$

# 7.6 STOKE'S THEOREM

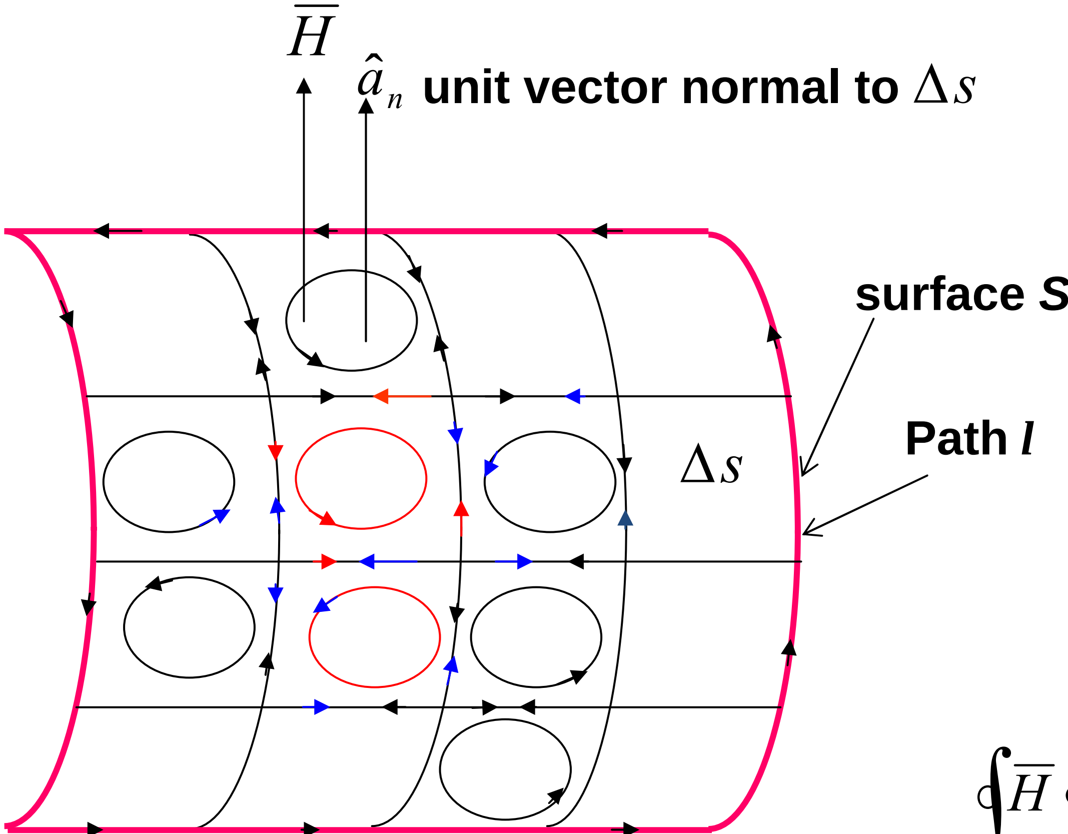
**Stoke's theorem states that the integral of the tangential component of a vector field $\overline{H}$ around *l* is equal to the integral of the normal component of curl $\overline{H}$ over *S.***

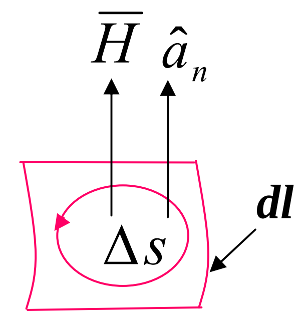
**In other word Stokes's theorem relates closed loop line integral $\overline{H} \cdot d\overline{l}$ to the surface integral $\left(\nabla \times \overline{H}\right) \cdot d\overline{s}$**

$$\oint_l \overline{H} \cdot d\overline{l} = \int_s \left(\nabla \times \overline{H}\right) \cdot d\overline{s}$$

**It can be shown as follow:**

**Consider an open surface *S* whose boundary is a closed surface *l***



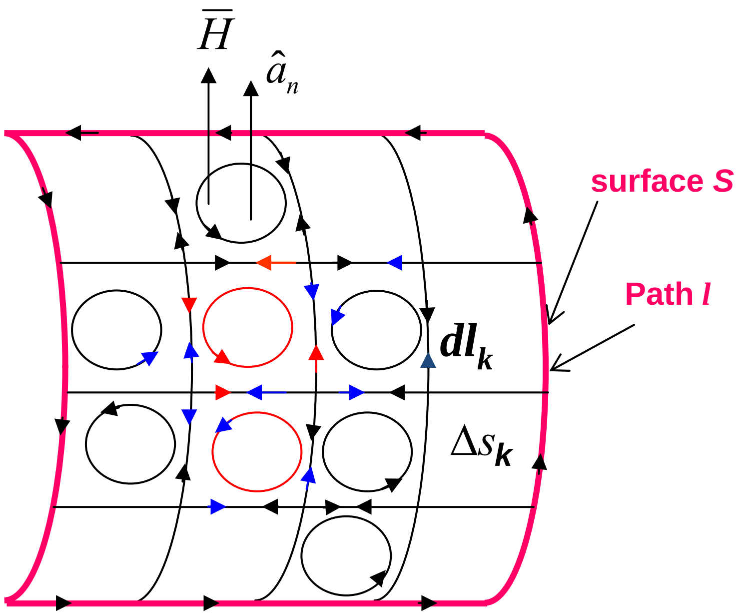




$$\frac{\oint \overline{H}\bullet\overline{dl}}{\Delta s} = \left(\nabla\times\overline{H}\right)\bullet\hat{a}_n$$

$$\oint \overline{H}\bullet\overline{dl} = \left(\nabla\times\overline{H}\right)\bullet\hat{a}_n\,\Delta s = \left(\nabla\times\overline{H}\right)\bullet\overline{\Delta s}$$

$$\sum_{k=1}^{m} \oint_{lk} \overline{H} \bullet \overline{dl_k} \approx \sum_{k=1}^{m} (\nabla \times \overline{H}) \bullet \overline{\Delta s_k}$$

**From the diagram it can be seen that the total integral of the surface $\Delta s$ enclosed by the loop inside the open surface *S* is zero since the adjacent loop is in the opposite direction. Therefore the total integral on the left side equation is the perimeter of the open surface *S*.**




***If*** $\Delta s_k \rightarrow 0$ **therefore** $m = \infty$

**Hence:**

$$\oint_{l} \overline{H} \bullet \overline{dl} = \int_{S} (\nabla \times H) \bullet \overline{ds}$$

**where loop *l* is the path that enclosed surface S and this equation is called Stoke's Theorem.**

**Ex. 7.8:** $\overline{H} = \hat{\phi}\left[\dfrac{Ir}{2\pi a^2}\right](A/m)$ **was found in an infinite conductor of radius a meter. Evaluate both side of Stoke's theorem to find the current flow in the conductor.**

***Solution:***

$$\oint_l \overline{H} \bullet d\overline{l} = \int_s (\nabla \times \overline{H}) \bullet d\overline{s}$$

$$\oint_l \left( \hat{\phi} \frac{Ir}{2\pi a^2} \right)\Bigg|_{r=a} \bullet \left(\hat{\phi} a\, d\phi\right) = \int \left( \nabla \times \hat{\phi} \frac{Ir}{2\pi a^2} \right) \bullet \hat{z}\, r\, dr\, d\phi$$

$$\int_0^{2\pi} \frac{I}{2\pi} d\phi = \int_0^a \int_0^{2\pi} \left( \hat{z} \frac{I}{\pi a^2} \right) \bullet \hat{z}\, r\, dr\, d\phi$$

$$I = I$$

$$\nabla \times \overline{H} = \frac{\hat{z}}{r_c}\left[\frac{\partial}{\partial r_c}\left(r_c \frac{I r_c}{2\pi a^2}\right)\right] = \frac{\hat{z}}{r_c}\left(\frac{2 r_c I}{2\pi a^2}\right) = \hat{z}\frac{I}{\pi a^2}$$

# 7.7 MAGNETIC FLUX DENSITY

**Magnetic field intensity :** $\overline{B} = \mu_o \overline{H} \quad \text{Teslas } (Wb/m^2)$

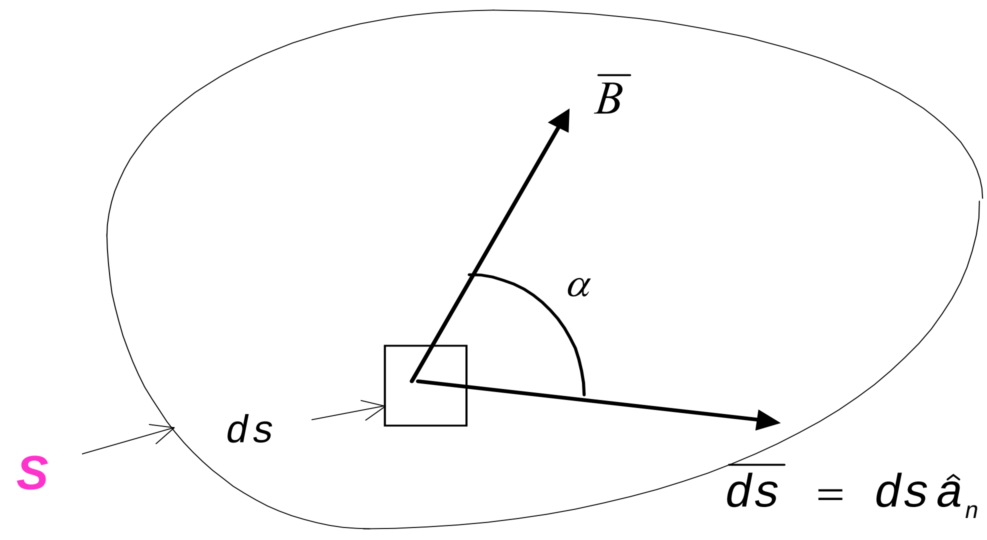
**where** $\mu_o = 4\pi \times 10^{-7} \ H/m$ **permeability of free space**

**Magnetic flux :** $\psi_m = \int_s \overline{B} \bullet \overline{ds}$ **that passes through the surface S.**

$$d\Psi_m = |\mathbf{ds}|\,|\overline{B}|\,|\cos\alpha = \overline{B} \bullet \overline{ds}$$

**Hence:**

$$\Psi_m = \int_s \overline{B} \bullet d\overline{s}$$

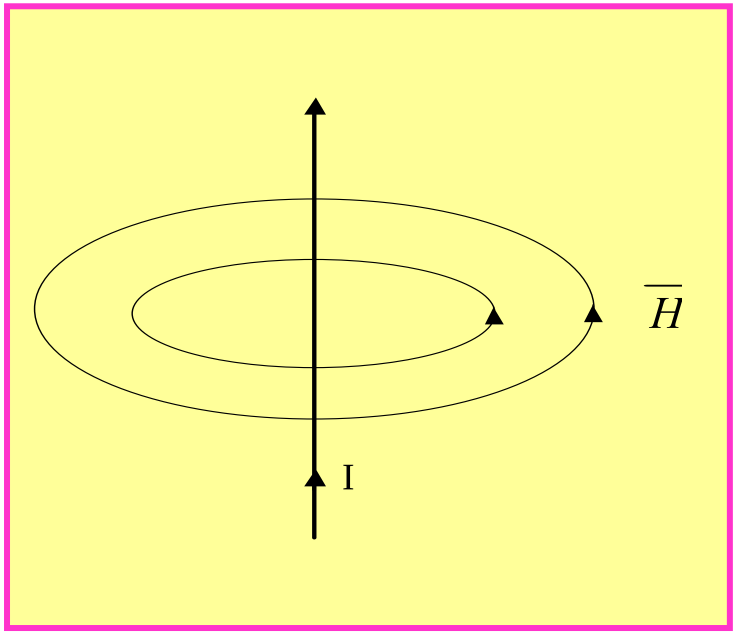


**In magnetics, magnet poles have not been isolated:**

$$\Psi_m = \oint_s \overline{B} \bullet d\overline{s} = 0 \quad (Wb)$$

$$\oint_s \overline{B} \bullet d\,\overline{s} = \int_v \left(\nabla \bullet \overline{B}\right) dv = 0$$

$$\nabla \bullet \overline{B} = 0$$




**4th. Maxwell's equation for static fields.**

**Ex. 7.9: For** $\overline{H} = \hat{\phi} 10^3 r$ **(Am$^{-1}$), find the** $\psi_m$ **that passes through a plane surface by, ($\phi = \pi/2$), ($2 \le r \le 4$), and ($0 \le z \le 2$).**

*Solution:*

$$\Psi_m = \int_s \overline{B} \bullet \overline{ds} = \int_0^2 \int_2^4 \left(\mu_o \hat{\phi} 10^3 r\right) \bullet \hat{\phi}\left(drdz\right)$$

$$= \mu_o 10^3 \int_0^2 \int_2^4 rdrdz = \mu_o 10^3 (12)$$

$$= 150.8 \text{ x } 10^{-4} \quad \text{Wb}$$

# 7.8 MAXWELL'S EQUATIONS

| POINT FORM | INTEGRAL FORM |
|---|---|
| $\nabla \bullet \overline{D} = \rho_v$ | $\int_v \nabla \bullet \overline{D}\, dv = \oint_s \overline{D} \bullet d\overline{s} = \int_v \rho_v\, dv = Q_{enc}$ |
| $\nabla \times \overline{E} = 0$ | $\int_s \left(\nabla \times \overline{E}\right) \bullet d\overline{s} = \oint_l \overline{E} \bullet d\overline{l} = 0$ |
| $\nabla \times \overline{H} = \overline{J}$ | $\int_s \left(\nabla \times \overline{H}\right) \bullet d\overline{s} = \oint_l \overline{H} \bullet d\overline{l} = \int_s \overline{J} \bullet d\overline{s} = I_{enc}$ |
| $\nabla \bullet \overline{B} = 0$ | $\int_v \nabla \bullet \overline{B}\, dv = \oint_s \overline{B} \bullet d\overline{s} = 0$ |

**Electrostatic fields :** $\overline{D} = \varepsilon \overline{E}$

**Magnetostatic fields:** $\overline{B} = \mu \overline{H}$

# 7.9 VECTOR MAGNETIC POTENTIAL

**To define vector magnetic potential, we start with:**

$$\oint_s \overline{B} \bullet d\overline{s} = 0$$

**=> magnet poles have not been isolated**

**Using divergence theorem:**

$$\oint \overline{B} \bullet \overline{ds} = \int_v \nabla \bullet \overline{B}\, dv = 0$$

**<=>**

$$\nabla \bullet \overline{B} = 0$$

**From vector identity:**

$$\nabla \bullet \left(\nabla \times \overline{A}\right) = 0$$

**where** $\overline{A}$ **is any vector.**

**Therefore from Maxwell and identity vector, we can defined if** $\overline{A}$ **is a vector magnetic potential, hence:**

$$\overline{B} = \nabla \times \overline{A}$$

# SUMMARY

Maxwell's equations

$$\oint_l \overline{H} \cdot d\overline{l} = \int_s \overline{J} \cdot d\overline{s} = \int_s \left(\nabla \times \overline{H}\right) \cdot d\overline{s} = I_{en}$$

Stoke's theorem

Ampere's circuital law

Maxwell's equations

$$\psi_m = \oint_s \overline{B} \cdot d\overline{s} = 0 = \int_v \nabla \cdot \overline{B}\, dv = 0$$

Divergence theorem

Gauss's law

Magnetic flux lines close on themselves

(Magnet poles cannot be isolated)