CHAPTER 6

LAPLACE'S EQUATION, POISSON'S EQUATION AND UNIQUENESS THEOREM

6.0 LAPLACE'S AND POISSON'S EQUATIONS AND UNIQUENESS THEOREM

- ***In realistic electrostatic problems, one seldom knows the charge distribution – thus all the solution methods introduced up to this point have a limited use.***

- ***These solution methods will not require the knowledge of the distribution of charge.***

6.1 LAPLACE'S AND POISSON'S EQUATIONS

To derive Laplace's and Poisson's equations , we start with Gauss's law in point form :

$$\nabla \bullet \overline{D} = \nabla \bullet \varepsilon \overline{E} = \rho_v \qquad \textbf{(1)}$$

Use gradient concept :

$$\overline{E} = -\nabla V \qquad \textbf{(2)}$$

$$\nabla \bullet \left[\varepsilon\left(-\nabla V\right)\right] = \rho_v$$

$$\nabla \bullet \nabla V = -\frac{\rho_v}{\varepsilon} \qquad \textbf{(3)}$$

Operator :

$$\nabla \bullet \nabla = \nabla^2 \qquad \textbf{(4)}$$

Hence :

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon} \quad V/m^2 \qquad \textbf{(5)}$$

=> Poisson's equation

is called Poisson's equation applies to a homogeneous media.

When the free charge density $\rho_v = 0$

$$\nabla^2 V = 0 \quad V/m^2$$ ***(6)*** ***=> Laplace's equation***

In rectangular coordinate :

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

6.2 UNIQUENESS THEOREM

Uniqueness theorem states that for a V solution of a particular electrostatic problem to be unique, it must satisfy two criterion :

(i) Laplace's equation

(ii) Potential on the boundaries

Example : ***In a problem containing two infinite and parallel conductors, one conductor in $z = 0$ plane at $V = 0$ Volt and the other in the $z = d$ plane at $V = V_0$ Volt, we will see later that the V field solution between the conductors is $V = V_0 z / d$ Volt.***

This solution will satisfy Laplace's equation and the known boundary potentials at $z = 0$ and $z = d$.

Now, the V field solution $V = V_0(z + 1) / d$ will satisfy Laplace's equation but will not give the known boundary potentials and thus is not a solution of our particular electrostatic problem.

Thus, $V = V_0 z / d$ Volt is the only solution (UNIQUE SOLUTION) of our particular problem.

6.3 SOLUTION OF LAPLACE'S EQUATION IN ONE VARIABLE

Ex.6.1: ***Two infinite and parallel conducting planes are separated d meter, with one of the conductor in the z = 0 plane at V = 0 Volt and the other in the z = d plane at V = V_0 Volt. Assume $\rho_v = 0$ and $\varepsilon = 2\varepsilon_0$ between the conductors.***

Find : (a) V in the range 0 < z < d ; (b) $\overline{E}$ between the conductors ;

(c) $\overline{D}$ between the conductors ; (d) D_n on the conductors ; (e) ρ_s on the conductors ; (f) capacitance per square meter.

Solution :

(a) Since $\rho_v = 0$ and the problem is in rectangular form, thus

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \qquad (1)$$

We note that V will be a function of z only V = V(z) ; thus :

$$\nabla^2 V = \frac{\partial^2 V}{\partial z^2} = 0 \qquad \textbf{(2)}$$

$$\frac{d^2 V}{dz^2} = \frac{d^2}{dz}\left(\frac{V}{dz}\right) = 0 \qquad \textbf{(3)}$$

Integrating twice :

$$\frac{dV}{dz} = A \qquad \textbf{(4)}$$

$$V = Az + B \qquad \textbf{(5)}$$

where A and B are constants and must be evaluated using given potential values at the boundaries :

$$V\big|_{z=0} = B = 0 \qquad \textbf{(6)}$$

$$V\big|_{z=d} = Ad = V_0$$

$$\rightarrow A = V_0 / d \qquad \textbf{(7)}$$

Substitute (6) and (7) into general equation (5) :

$$\therefore V = \frac{V_0}{d} z \quad (V) \qquad 0 < z < d$$

(b)

$$\overline{E} = -\nabla V = -\left(\hat{x} \frac{\partial V}{\partial x} + \hat{y} \frac{\partial V}{\partial y} + \hat{z} \frac{\partial V}{\partial z} \right)$$

$$= -\hat{z} \frac{\partial V}{\partial z} = -\hat{z} \frac{V_0}{d} \quad (V/m)$$

(c)

$$\overline{D} = \varepsilon \overline{E} = -\hat{z} \frac{2\varepsilon_0 V_0}{d} \qquad (C/m^2)$$

(d) Surface charge :

$$\rho_s\big|_{z=0} = \overline{D} \bullet \hat{n} = -\hat{z}\,\frac{2\varepsilon_0 V_0}{d} \bullet \hat{z}$$

$$= -\frac{2\varepsilon_0 V_0}{d}$$

$$\rho_s\big|_{z=d} = \overline{D} \bullet \hat{n} = -\hat{z}\,\frac{2\varepsilon_0 V_0}{d} \bullet (-\hat{z})$$

$$= +\frac{2\varepsilon_0 V_0}{d} \qquad (C/m^2)$$

(e) Capacitance :

$$C = Q/V_{ab}$$

$$= \frac{\left|\rho_s ds\right|}{V_0}$$

$$\therefore C/m^2 = \frac{\left|\rho_s\right|}{V_0}$$

$$= \frac{2\varepsilon_0 V_0/d}{V_0} = \frac{\varepsilon}{d} \qquad (F/m^2)$$

***Ex.6.2:* Two infinite length, concentric and conducting cylinders of radii a and b are located on the z axis. If the region between cylinders are charged free and $\varepsilon = 3\varepsilon_0$, V = V_0 (V) at a, V = 0 (V) at b and b > a. Find the capacitance per meter length.**

***Solution :* Use Laplace's equation in cylindrical coordinate :**

and V = f(r) only :

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

and V = f(r) only :

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) = 0$$

$$\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) = 0$$

$$\left(r\frac{\partial V}{\partial r}\right) = A$$

$$\frac{\partial V}{\partial r} = \frac{A}{r}$$

$$V = A\ln r + B \qquad \textbf{(1)}$$

Boundary condition :

$$V\Big|_{r=a} = V_0 = A\ln a + B$$

$$V\Big|_{r=b} = 0 = A\ln b + B$$

$$V = A\ln r + B \quad \textbf{(1)}$$

Solving for A and B :

$$A = \frac{V_0}{\ln(a/b)} \quad ; \quad B = \frac{-V_0 \ln b}{\ln(a/b)}$$

Substitute A and B in (1) :

$$\therefore V = \frac{V_0 \ln(b/r)}{\ln(b/a)} \quad ; \quad a < r < b$$

$$\overline{E} = -\nabla V = -\hat{r}\frac{\partial V}{\partial r} = \frac{V_0}{r\ln(b/a)}\hat{r}$$

$$\overline{D} = \varepsilon\overline{E} = \frac{\varepsilon V_0}{r\ln(b/a)}\hat{r}$$

$$\therefore V = \frac{V_0 \ln(b/r)}{\ln(b/a)}$$

Surface charge densities:

$$\rho_s\Big|_{r=a} = \overline{D}\cdot\hat{r} = \frac{\varepsilon V_0}{a\ln(b/a)}$$

$$\rho_s\Big|_{r=b} = \overline{D}\cdot -\hat{r} = -\frac{\varepsilon V_0}{b\ln(b/a)}$$

Line charge densities :

$$\rho_\ell\Big|_{r=a} = \rho_s\Big|_{r=a}(2\pi a) = \frac{2\pi\varepsilon V_0}{\ln(b/a)}$$

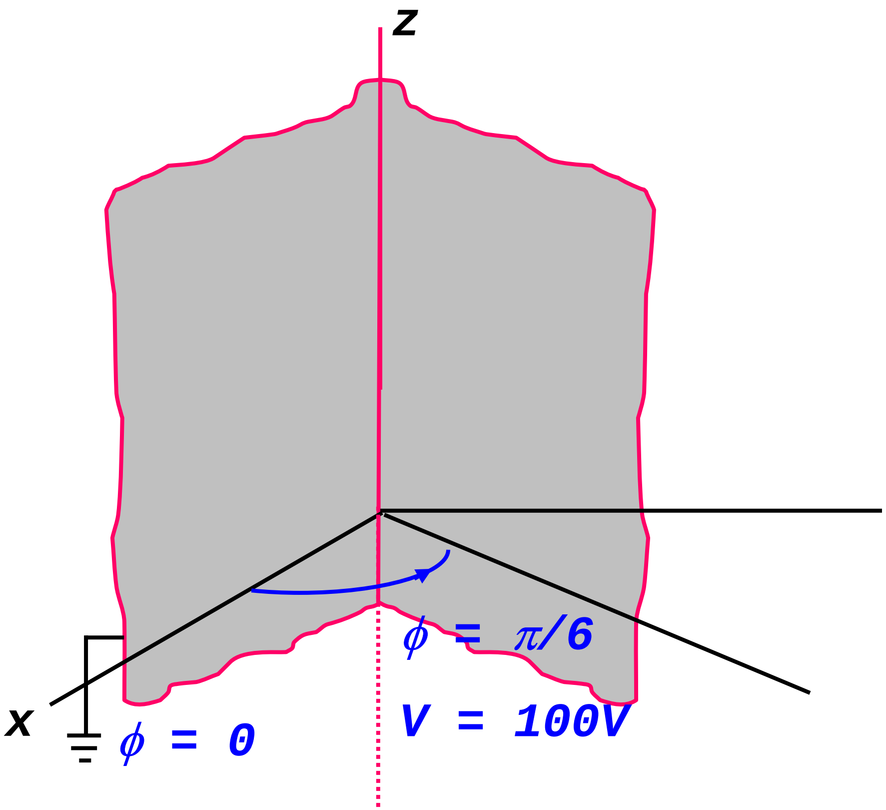
$$\rho_\ell\Big|_{r=b} = \rho_s\Big|_{r=b}(2\pi b) = -\frac{2\pi\varepsilon V_0}{\ln(b/a)}$$

Capacitance per unit length:

$$C = \frac{Q}{V_{ab}} = \frac{|\rho_\ell d\ell|}{V_o}$$

$$C/m = \frac{|\rho_\ell|}{V_0} = \frac{2\pi\varepsilon}{\ln(b/a)} \quad (F/m)$$

Ex.6.3: ***Two infinite conductors form a wedge located at $\phi = 0$ and $\phi = \pi/6$ is as shown in the figure below. If this region is characterized by charged free. Find V and $\bar{E}$. Assume V = 0 V at $\phi = 0$ and $V = 100\,V$ at $\phi = \pi/6$.***



Solution : ***V = f (ϕ) in cylindrical coordinate :***

$$\nabla^2 V = \frac{1}{r^2}\frac{d^2V}{d\phi^2} = 0$$

$$\frac{d^2V}{d\phi^2} = 0$$

$$\frac{dV}{d\phi} = A$$

$$V = A\phi + B$$

Boundary condition :

$$V\big|_{\phi=0} = 0 = B$$

$$V\big|_{\phi=\pi/6} = 100 = A(\pi/6)$$

$$A = 600/\pi$$

Hence :

$$V = \frac{600}{\pi}\phi$$

$$\overline{E} = -\nabla V = -\frac{1}{r}\frac{dV}{d\phi}\hat{\phi}$$

$$= -\frac{600}{\pi r}\hat{\phi}$$

for region :

$$0 \le \phi \le \pi/6$$

Ex.6.4: ***Two infinite concentric conducting cone located at $\theta = \pi/10$ and $\theta = \pi/6$. The potential V = 0 V at $\theta = \pi/10$ and V = 50 V at $\theta = \pi/6$. Find V and $\overline{E}$ between the two conductors.***

Solution : ***V = f (θ) in spherical coordinate :***

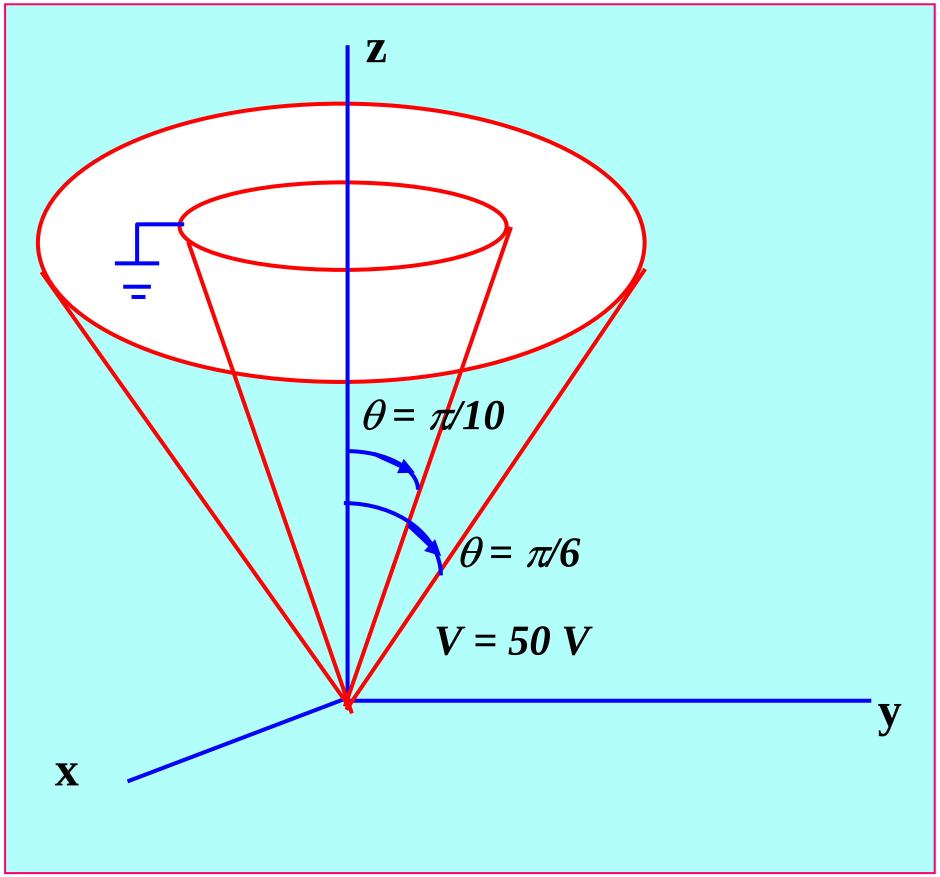
$$\nabla^2 V = \frac{1}{r^2 \sin\theta}\frac{d}{d\theta}\left[\sin\theta \frac{dV}{d\theta}\right] = 0$$

$$\frac{d}{d\theta}\left[\sin\theta \frac{dV}{d\theta}\right] = 0$$

$$\sin\theta \frac{dV}{d\theta} = A$$

$$\frac{dV}{d\theta} = \frac{A}{\sin\theta}$$

$$V = A\ln\left(\tan\theta/2\right) + B$$



Using : $\int \frac{d\theta}{\sin\theta} = \ln\left(\tan\theta/2\right)$

$$V = A\ln\left(\tan\theta/2\right) + B$$

Boundary condition :

$$V\Big|_{\theta=\pi/10} = 0 = A\ln\left(\tan\pi/20\right) + B$$

$$V\Big|_{\theta=\pi/6} = 50 = A\ln\left(\tan\pi/12\right) + B$$

Solving for A and B :

$$A = \frac{50}{\ln\left(\dfrac{\tan\pi/12}{\tan\pi/20}\right)} \;;\; B = -\frac{50\ln\tan\pi/20}{\ln\left(\dfrac{\tan\pi/12}{\tan\pi/20}\right)}$$

Hence at region : $\theta/10 \le \theta \le \theta/6$

$$V = \frac{50}{\ln\left(\dfrac{\tan\pi/12}{\tan\pi/20}\right)} \ln\frac{\tan\theta/2}{\tan\pi/20}$$

$$= 95.1\ln\left(\frac{\tan\theta/2}{0.1584}\right)$$

and

$$\overline{E} = -\nabla V = \frac{1}{r}\frac{dV}{d\theta}\hat{\theta}$$

$$= -\frac{95.1}{r\sin\theta}\hat{\theta}$$

6. 4 SOLUTION FOR POISSON'S EQUATION

When the free charge density $\rho_v \neq 0$

Ex.6.5: ***Two infinite and parallel conducting planes are separated d meter, with one of the conductor in the x = 0 plane at V = 0 Volt and the other in the x = d plane at V = V_0 Volt. Assume*** $\rho_v \neq 0$ ***and*** $\varepsilon = 4\varepsilon_0$ ***between the conductors.***

Find : (a) V in the range 0 < x < d ; (b) $\overline{E}$ ***between the conductors***

Solution :

V = f(x) :

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon}$$

$$\frac{d^2 V}{dx^2} = -\frac{\rho_0}{\varepsilon}$$

$$\frac{dV}{dx} = -\frac{\rho_0}{\varepsilon}x + A$$

$$V = -\frac{\rho_0}{\varepsilon}\frac{x^2}{2} + Ax + B$$

Boundary condition :

$$V\Big|_{x=0} = 0 = B$$

$$V\Big|_{x=d} = V_0 = -\frac{\rho_0}{\varepsilon}\frac{d^2}{2} + Ad$$

$$A = \frac{V_0}{d} + \frac{\rho_0}{\varepsilon}\frac{d}{2}$$

$$V = -\frac{\rho_0}{\varepsilon}\frac{x^2}{2} + Ax + B$$

In region : $0 \le x \le d$

$$V = \frac{\rho_0}{\varepsilon}\frac{x}{2}(d - x) + \frac{V_0}{d}x \quad ; \quad \overline{E} = -\frac{dV}{dx}\hat{x}$$

$$= -\left[\frac{V_0}{d} + \frac{\rho_0}{\varepsilon}\left(\frac{d}{2} - x\right)\right]\hat{x}$$

***Ex.6.6:* Repeat Ex.6.5 with** $\rho_v = 0$ and $\varepsilon_r = 1 + x$

Solution :

$$\nabla \bullet \overline{D} = \rho_v$$

$$\nabla \bullet \varepsilon \overline{E} = 0$$

$$\frac{d}{dx}(1 + x)\varepsilon_0 \overline{E} = 0$$

$$\frac{d}{dx}(1 + x)(-\nabla V) = 0$$

$$(1 + x)\left(-\frac{dV}{dx}\right) = A$$

$$\left(-\frac{dV}{dx}\right) = \frac{A}{1 + x}$$

$$V = -A\ln(1 + x) + B$$

Boundary condition :

$$V\Big|_{x=0} = 0 = B$$

$$V\Big|_{x=d} = V_0 = -A\ln(1+d)$$

$$\rightarrow A = -\frac{V_0}{\ln(1+d)}$$

In region : $0 \le x \le d$

$$V = -A\ln(1+x) + B$$

$$V = V_0\frac{\ln(1+x)}{\ln(1+d)}$$

$$\overline{E} = -\frac{dV}{dx}\hat{x} = -\frac{V_0}{(1+x)\ln(1+d)}\hat{x}$$