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ELECTROMAGNETICS THEORY
(SEE 2523)

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CHAPTER 4

POTENTIAL DIFFERENCE, GRADIENT & ENERGY IN ELECTRIC FIELD

INTRODUCTION

- Electric scalar potential V is used to obtain $\overline{E}$.
- Easier compare with Coulomb's law and Gauss law because:

1) Scalar quantity
2) Using differentiation

4.1 ENERGY & POTENTIAL

❖ Consider to move a point charge *Q* from point *B* to point *A* in an electric field, $\overline{E}$ as shown in Fig. 4.1.

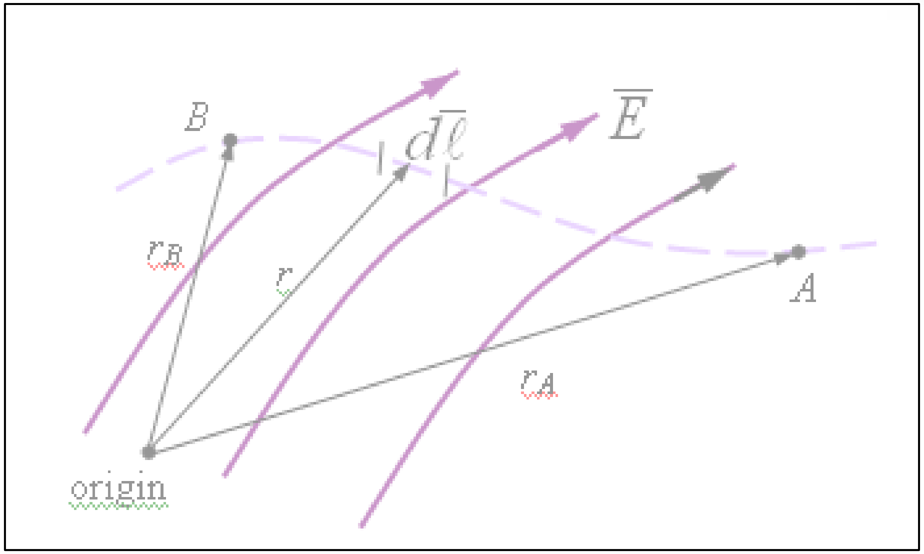

Fig. 4.1

❖ From Coulomb's law, the force on Q is $\overline{F} = Q\overline{E}$, so that the work done in displacing the charge by $d\overline{\ell}$ is

$$dW = -\overline{F} \cdot d\overline{\ell} = -Q\overline{E} \cdot d\overline{\ell} \qquad (1)$$

❖ The negative sign indicates that the work is being done by an external agent.

❖ From Eq. 1, the work done in moving Q from A to B is

$$W = -Q\int_{B}^{A} \overline{E} \cdot d\overline{\ell} \qquad (2)$$

❖ Dividing Eq. 2 by Q gives the potential difference, V_{AB} between points *A* and *B*. Thus,

$$V_{AB} = W/Q = -\int_{B}^{A} \overline{E} \cdot d\overline{\ell} \qquad (3)$$

The potential difference is defined as total work done in displacing a positive charge *Q* from point *B* to point *A* in an electric field $\overline{E}$.

NOTE THAT !

In determining V_{AB}, B is the initial point while A is the final point

$d\bar{\ell}$ is always positive even though in opposite path

V_{AB} is independent of the path taken

Potential is a scalar quantity and its unit is volts (V)

❖ Refer Fig. 4.2.

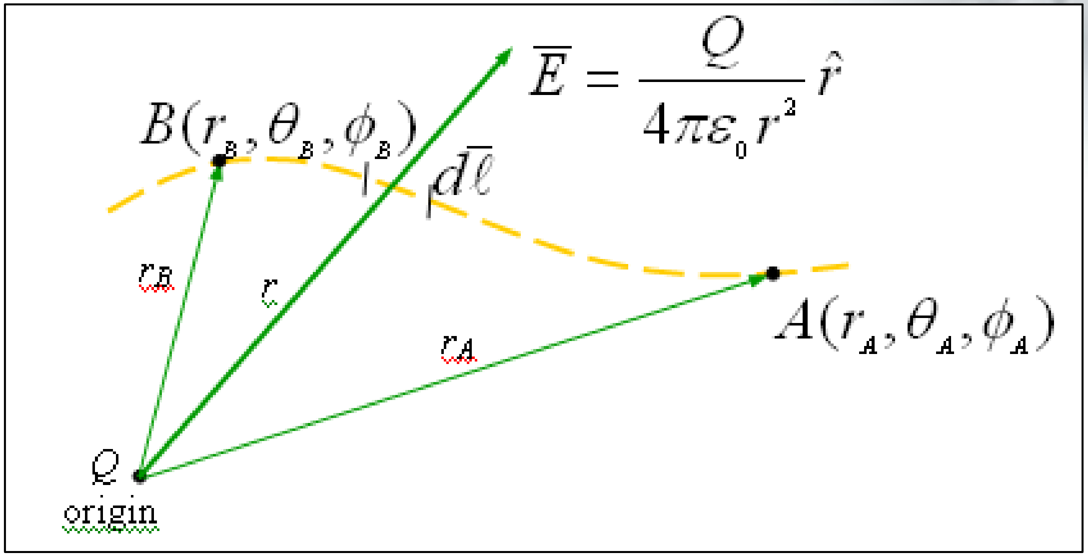

Fig. 4.2

❖ Due to a point charge Q located at the origin, then

$$\overline{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r}$$

❖ $d\overline{\ell}$ in spherical coordinate is

$$d\overline{\ell} = \hat{r}dr + \hat{\theta}rd\theta + \hat{\phi}r\sin\theta d\phi$$

$$\therefore V_{AB} = -\int_B^A \overline{E} \cdot d\overline{\ell}$$

$$= -\int_B^A \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r} \cdot \left(\hat{r}dr + \hat{\theta}rd\theta + \hat{\phi}r\sin\theta d\phi\right)$$

$$= -\frac{Q}{4\pi\varepsilon_0}\int_B^A \frac{dr}{r^2}$$

$$= \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r_A} - \frac{1}{r_B}\right) \qquad (4)$$

❖ From Eq. 4, note that $\overline{E}$ points in the radial direction.

❖ Any contribution from displacement $d\overline{\ell}$ in the $\hat{\theta}$ or $\hat{\phi}$ direction is wiped out by the dot product $\overline{E} \cdot d\overline{\ell}$.

❖ Potential difference is independent of the path between *A* and *B*.

$$\therefore V_{AB} = -V_{BA}$$

$$V_{BA} + V_{AB} = \oint_{\ell} \overline{E} \cdot d\overline{\ell} = 0 \qquad (5)$$

- Eq. 5 shows that the line integral of $\overline{E}$ along a closed path as shown in Fig. 4.3 must be zero.
- Means no net work is done in moving a charge along a closed path.
- This is said to be conservative.
- This conservative field only true for static electric field.

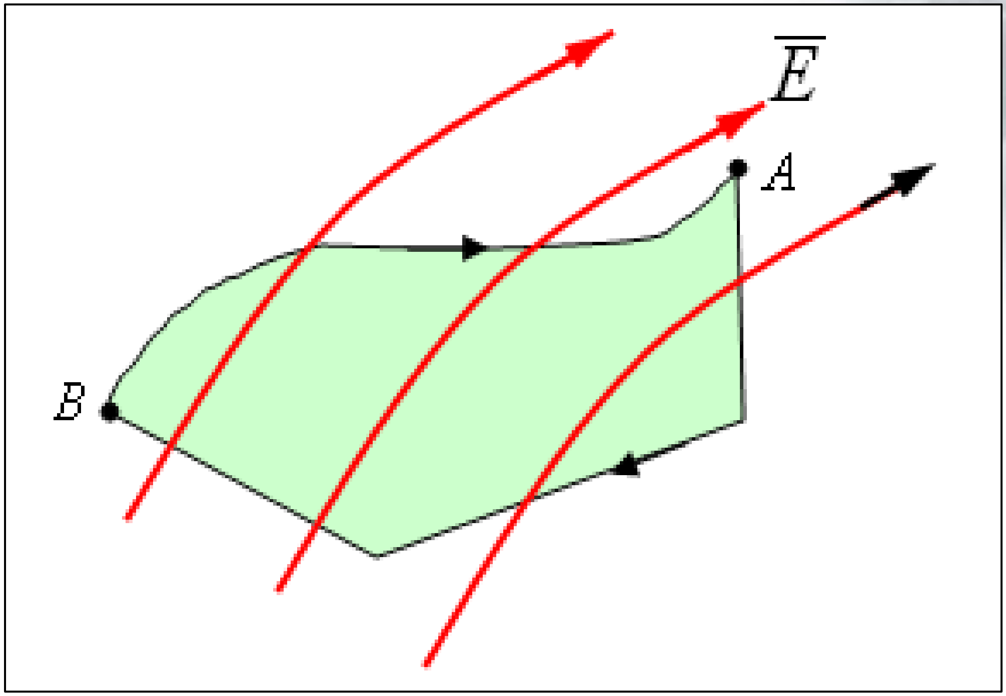

Fig. 4.3

❖ Using Stoke's Theorem, the equation becomes,

$$\oint_{\ell} \overline{E} \cdot d\overline{\ell} = \int_{S} (\nabla \times \overline{E}) \cdot d\overline{s} = 0 \quad (6)$$

or

$$\nabla \times \overline{E} = 0 \quad (7)$$

❖ The Eq. 7 is referred to Maxwell's Equation while the Eq. 5 is the integral form.

❖ The Eq. 4 can be written as

$$V_{AB} = V_A - V_B \qquad (8)$$

where V_A and V_B are the potentials at *A* and *B*, respectively.

❖ In problem solving, it is customary to choose infinity as reference.

❖ Means the potential at infinity is zero.

❖ Assume point B is infinity, $V_B = 0$ $(r_B \to \infty)$ then Eq.4 becomes

$$V_{A\infty} = \frac{Q}{4\pi\varepsilon_0 r_A} \qquad (9)$$

or

$$V = \frac{Q}{4\pi\varepsilon_0 r} \qquad (10)$$

❖ From Eq. 10, at certain distance r, potential at any point on the spherical surface is same.

❖ No work is done in moving a unit charge around on *equipotential surface* because no potential difference between any two points on this surface.

❖ In Eq.10, the reference point is at infinity ($V = 0$).

❖ If other point is selected as the reference point, Eq. 4 becomes

$$V = \frac{Q}{4\pi\varepsilon_0 r} + C \qquad (11)$$

where C is a constant that is determined at the chosen point of reference.

4.2 THE POTENTIAL OF A SYSTEM OF CHARGES

❖ Consider there are 3 point charges as shown in Fig. 4.5.

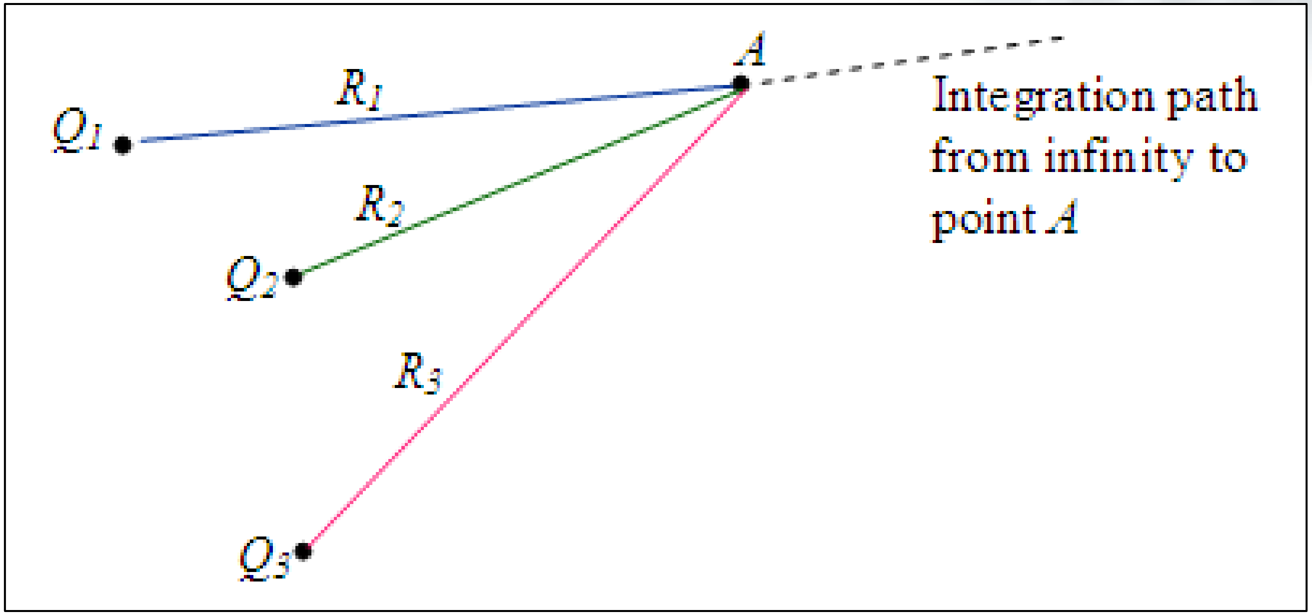

Fig. 4.5

❖ Absolute potential at point *A* is given by

$$V_A = -\int_{\infty}^{A} \overline{E}_1 \cdot d\overline{\ell} - \int_{\infty}^{A} \overline{E}_2 \cdot d\ell - \int_{\infty}^{A} \overline{E}_3 \cdot d\overline{\ell} \qquad (12)$$

where E_1, E_2 and E_3 refer to the field caused by the 3 charges.

❖ Using $\overline{E}$ for a point charge, the potential at point *A* becomes

$$V_A = \frac{Q_1}{4\pi\varepsilon_0 R_1} + \frac{Q_2}{4\pi\varepsilon_0 R_2} + \frac{Q_3}{4\pi\varepsilon_0 R_3} = \sum_{i=1}^{3} \frac{Q_i}{4\pi\varepsilon_0 R_i} \qquad (13)$$

❖ Eq. 13 can be written as

$$V_A = V_1 + V_2 + V_3 \qquad (14)$$

where V_1, V_2 and V_3 are the absolute potential at point A caused by the 3 charges.

❖ The general form of any charge distribution is given by

$$V_A = \int \frac{dQ}{4\pi\varepsilon_0 R} \qquad (15)$$

❖ Potential for line, surface and volume charge are given below.

$$V_A = \int \frac{\rho_\ell \, d\ell}{4\pi\varepsilon_0 R}$$

$$V_A = \int \frac{\rho_s \, ds}{4\pi\varepsilon_0 R}$$

$$V_A = \int \frac{\rho_v \, dv}{4\pi\varepsilon_0 R}$$

❖ 2 ways to determine the absolute potential.

Field

Charge distribution

4.3 POTENTIAL GRADIENT

❖ From the definition of potential

$$V = -\int \overline{E} \cdot d\overline{\ell}$$

$$dV = -\overline{E} \cdot d\overline{\ell}$$

$$= -E_x dx - E_y dy - E_z dz$$

❖ *dV* is also equal to

$$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz$$

❖ Comparing the two expression for *dV*, then

$$E_x = -\frac{\partial V}{\partial x}, \quad E_y = -\frac{\partial V}{\partial y}, \quad E_z = -\frac{\partial V}{\partial z}$$

$$\overline{E} = -\frac{\partial V}{\partial x}\hat{x} - \frac{\partial V}{\partial y}\hat{y} - \frac{\partial V}{\partial z}\hat{z}$$

- In other way

$$\overline{E} = -\nabla V$$

- The electric field intensity is opposite with the gradient of *V*.
- The negative sign shows that the direction of $\overline{E}$ is opposite to the direction in which *V* increases.
- $\overline{E}$ is directed from higher to lower levels of *V*.
- The gradient is a vector quantity.

❖ Equation of ∇V for the 3 coordinate systems:

$$\nabla V = \frac{\partial V}{\partial x}\hat{x} + \frac{\partial V}{\partial y}\hat{y} + \frac{\partial V}{\partial z}\hat{z}$$

$$\nabla V = \frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial \phi}\hat{\phi} + \frac{\partial V}{\partial z}\hat{z}$$

$$\nabla V = \frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{\phi}$$

4.4 ENERGY DENSITY IN THE ELECTRIC FIELD

- Energy in $\bar{E}$ is a total work is being done to build a charge system.
- Refer Fig. 4.8

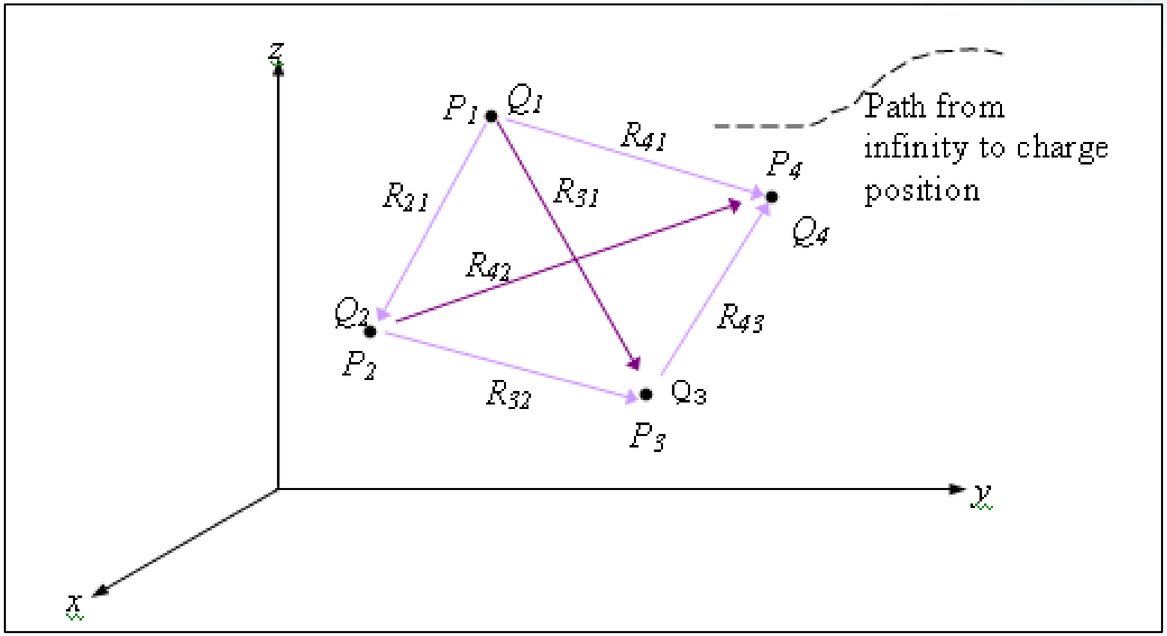

Fig. 4.8

- Start by visualizing an empty universe.

- Bringing a charge Q_1 from infinity to P_1 requires no work because there is no field present.

$$W_1 = 0$$

- The positioning of Q_2 at P_2, requires an amount of work because there is field present due to Q_1.

$$W_2 = V_{21} Q_2$$

- V_{21} is a potential at P_2 due to Q_1.
- The first subscript indicates the location and the second subscript the source.
- Work to position Q_3 and Q_4.

$$W_3 = V_{31}Q_3 + V_{32}Q_3$$
$$W_4 = V_{41}Q_4 + V_{42}Q_4 + V_{43}Q_4$$

- Total positioning work.

$$W_E = V_{21}Q_2 + V_{31}Q_3 + V_{32}Q_3 + V_{41}Q_4 + V_{42}Q_4 + V_{43}Q_4$$

❖ Example:

$$V_{31}Q_3 = Q_3 \frac{Q_1}{4\pi\varepsilon_0 R_{13}} = Q_1 \frac{Q_3}{4\pi\varepsilon_0 R_{31}}$$

where R_{13} and R_{31} represent distance between Q_1 and Q_3. It might be written as $V_{13}Q_1$.

❖ If each term of the energy expression is replaced by its equal, then

$$W_E = V_{12}Q_1 + V_{13}Q_1 + V_{23}Q_2 + V_{14}Q_1 + V_{24}Q_2 + V_{34}Q_3$$

❖ Adding the two energy expression, obtain

$$2W_E = Q_1\left(V_{12} + V_{13} + V_{14}\right) + Q_2\left(V_{21} + V_{23} + V_{24}\right) + Q_3\left(V_{31} + V_{32} + V_{34}\right) + Q_4\left(V_{41} + V_{42} + V_{43}\right)$$

❖ $\left(V_{12} + V_{13} + V_{14}\right)$ is a absolute potential at P_1 due to all the charges at the point where this combined potential is being found.

❖ The expression is equal to V_1,

$$2W_E = Q_1V_1 + Q_2V_2 + Q_3V_3 + Q_4V_4$$

$$W_E = \frac{1}{2}\sum_{k=1}^{n} Q_k V_k$$

❖ Instead of point charges, the equation becomes,

$$W_E = \frac{1}{2}\int_v V\rho_v dv$$

$$W_E = \frac{1}{2}\int_s V\rho_s ds$$

$$W_E = \frac{1}{2}\int_\ell V\rho_\ell d\ell$$

❖ The energy equation can be obtained in term of $\overline{E}$ and $\overline{D}$.

❖ Using $\rho_v = \nabla \cdot \overline{D}$ and vector identity $\nabla \cdot \left(f\overline{A} \right) = f\left(\nabla \cdot \overline{A} \right) + \overline{A} \cdot \left(\nabla f \right)$

❖ Energy equation for volume charge becomes

$$W_E = \frac{1}{2}\int_V V\left(\nabla \cdot \overline{D}\right)dv$$
$$= \frac{1}{2}\int_V \left[\nabla \cdot \left(V\overline{D}\right) - \overline{D} \cdot \left(\nabla V\right)\right]dv$$

❖ By applying divergence theorem,

$$W_E = \frac{1}{2}\oint_S \left(V\overline{D}\right) \cdot d\overline{s} - \frac{1}{2}\int_V \overline{D} \cdot \left(\nabla V\right)dv$$

❖ The surface integral is equal to zero when $r \to \infty$ and replace $\overline{E} = -\nabla V$

$$W_E = \frac{1}{2}\int_v \overline{D} \cdot \overline{E}\, dv = \frac{1}{2}\int_v \varepsilon_0 \overline{E}^2 dv, \quad \text{(J)}$$

❖ In electromagnetic field theory, the energy of an electric field is stored in the field itself.

❖ So, the energy density in a field is given by

$$dW_E = \frac{1}{2}\overline{D} \cdot \overline{E}\, dv$$

or

$$\frac{dW_E}{dv} = \frac{1}{2}\overline{D} \cdot \overline{E} = \frac{1}{2}\varepsilon_0 \overline{E}^2, \quad \text{(J/m}^3\text{)}$$