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CHAPTER 10

PROPAGATION & REFLECTION OF PLANE WAVES

10.0 PROPAGATION & REFLECTION OF PLANE WAVES

Will discuss the effect of propagation of EM wave in four medium : Free space ; Lossy dielectric ; Lossless dielectric (perfect dielectric) and Conducting media.

Also will be discussed the phenomena of reflections at interface between different media.

Ex : EM wave is radio wave, TV signal, radar radiation and optical wave in optical fiber.

Three basics characteristics of EM wave :

- ***travel at high velocity***
- ***travel following EM wave characteristics***
- ***travel outward from the source***

These propagation phenomena for a type traveling wave called plane wave can be explained or derived by Maxwell's equations.

10.1 ELECTRIC AND MAGNETIC FIELDS FOR PLANE WAVE

From Maxwell's equations :

$$\nabla \times \overline{E} = -\frac{\partial \overline{B}}{\partial t} = -\mu \frac{\partial \overline{H}}{\partial t}$$

$$\nabla \times \overline{H} = \overline{\mathrm{J}} + \frac{\partial \overline{D}}{\partial t} = \overline{\mathrm{J}} + \varepsilon \frac{\partial \overline{E}}{\partial t}$$

$$\nabla \bullet \overline{D} = \rho_v$$

$$\nabla \bullet \overline{B} = 0$$

Assume the medium is free of charge : $\rho_v = 0, \overline{J} = 0$

$$\nabla \times \overline{E} = -\mu \frac{\partial \overline{H}}{\partial t} \qquad (1)$$

$$\nabla \times \overline{H} = \varepsilon \frac{\partial \overline{E}}{\partial t} \qquad (2)$$

$$\nabla \bullet \overline{D} = 0 \qquad (3)$$

$$\nabla \bullet \overline{B} = 0 \qquad (4)$$

From vector identity and taking the curl of (1)and substituting (1) and (2)

$$\nabla \times (\nabla \times \overline{E}) = \nabla(\nabla \bullet \overline{E}) - \nabla^2 \overline{E}$$

$$where \quad \nabla(\nabla \bullet \overline{E}) = 0$$

$$=> \nabla \times \left(-\mu \frac{\partial \overline{H}}{\partial t} \right) = -\nabla^2 \overline{E}$$

$$\rightarrow \nabla^2 \overline{E} = \mu \frac{\partial}{\partial t} (\nabla \times \overline{H})$$

$$\therefore \nabla^2 \overline{E} = \mu\varepsilon \frac{\partial^2 \overline{E}}{\partial t^2}$$

ie Helmholtz 's equation for electric field

$$\nabla^2 \overline{E} = \mu\varepsilon \frac{\partial^2 \overline{E}}{\partial t^2} \quad Vm^{-3}$$

Similarly in the same way, from vector identity and taking the curl of (2)and substituting (1) and (2)

$$\nabla^2 \overline{H} = \mu\varepsilon \frac{\partial^2 \overline{H}}{\partial t^2} \quad \mathrm{Am}^{-3}$$

In Cartesian coordinates :

$$\frac{\partial^2 \overline{E}}{\partial x^2} + \frac{\partial^2 \overline{E}}{\partial y^2} + \frac{\partial^2 \overline{E}}{\partial z^2} = \mu\varepsilon \frac{\partial^2 \overline{E}}{\partial t^2}$$

Assume that :

(i) Electric field only has x component

(ii) Propagate in the z direction

$$\frac{\partial^2 E_x}{\partial z^2} = \mu\varepsilon \frac{\partial^2 E_x}{\partial t^2}$$

$$\frac{\partial^2 E_x}{\partial z^2} = \mu\varepsilon \frac{\partial^2 E_x}{\partial t^2}$$

The solution for this equation :

$$E_x = E_x^+ \cos(\omega t - \beta z) + E_x^- \cos(\omega t + \beta z)$$

Incidence wave propagate in +z direction

Reflected wave propagate in -z direction

To find H field :

$$\nabla \times \overline{E} = -\mu \frac{\partial \overline{H}}{\partial t}$$

$$\nabla \times \overline{E} = \frac{\partial E_x}{\partial z}\hat{y} - \frac{\partial E_x}{\partial y}\hat{z}$$

$$= \left\{\beta E_x^+ \sin(\omega t - \beta z) - \beta E_x^- \sin(\omega t + \beta z)\right\}\hat{y}$$

On the right side equation :

$$-\mu \frac{\partial \overline{H}}{\partial t} = -\mu \left\{ \frac{\partial H_x}{\partial t}\hat{x} + \frac{\partial H_y}{\partial t}\hat{y} + \frac{\partial H_z}{\partial t}\hat{z} \right\}$$

Equating components on both side = y component

$$-\mu \frac{\partial H_y}{\partial t} = \left\{ \beta E_x^{+} \sin(\omega t - \beta z) - \beta E_x^{-} \sin(\omega t + \beta z) \right\}$$

$$-H_y = \int \frac{\beta E_x^{+}}{\mu} \sin(\omega t - \beta z) dt - \int \frac{\beta E_x^{-}}{\mu} \sin(\omega t + \beta z) dt$$

$$= -\frac{\beta}{\omega\mu} E_x^{+} \cos(\omega t - \beta z) + \frac{\beta}{\omega\mu} E_x^{-} \cos(\omega t + \beta z)$$

$$H_y = \frac{\beta}{\omega\mu} E_x^{+} \cos(\omega t - \beta z) - \frac{\beta}{\omega\mu} E_x^{-} \cos(\omega t + \beta z)$$

$$= H_y^{+} \cos(\omega t - \beta z) - H_y^{-} \cos(\omega t + \beta z)$$

Hence :

$$E_x = E_x^+ \cos(\omega t - \beta z) + E_x^- \cos(\omega t + \beta z)$$

$$H_y = H_y^+ \cos(\omega t - \beta z) - H_y^- \cos(\omega t + \beta z)$$

These equations of EM wave are called PLANE WAVE.

Main characteristics of EM wave :

(i) Electric field and magnetic field always perpendicular.

(ii) NO electric or magnetic fields component in the direction of propagation.

(iii) $\overline{E} \times \overline{H}$ ***will provides information on the direction of propagation.***

10.2 PLANE WAVE IN LOSSY DIELECTRICS – IMPERFECT DIELECTRICS

$$\sigma \neq 0\,;\, \mu = \mu_0 \mu_r\,;\, \varepsilon = \varepsilon_0 \varepsilon_r$$

Assume a media is charged free , ρ_v =0

$$\nabla \times \overline{H} = \overline{J} + \frac{\partial \overline{D}}{\partial t} = (\sigma + j\omega\varepsilon)\overline{E} \qquad (1)$$

$$\nabla \times \overline{E} = -\frac{\partial \overline{B}}{\partial t} = -j\omega\mu\overline{H} \qquad (2)$$

Taking the curl of (2) :

$$\nabla \times \nabla \times \overline{E} = -j\omega\mu\left(\nabla \times \overline{H}\right)$$

From vector identity :

$$\nabla\times\nabla\times\overline{A}=\nabla\left(\nabla\bullet\overline{A}\right)-\nabla^2\overline{A}$$

$$\nabla\left(\nabla\bullet\overline{E}\right)-\nabla^2\overline{E}=-j\omega\mu\left(\sigma+j\omega\varepsilon\right)\overline{E}$$

$$\nabla^2\overline{E}-j\omega\mu\left(\sigma+j\omega\varepsilon\right)\overline{E}=0$$

$$\nabla^2\overline{E}-\gamma^2\overline{E}=0$$

Where :

$$\gamma^2=j\omega\mu\left(\sigma+j\omega\varepsilon\right)=-\omega^2\mu\varepsilon+j\omega\mu\sigma \quad \textbf{(4)}$$

$$\gamma=\text{propagation constant}$$

Define :

$$\gamma=\alpha+j\beta$$

$$\gamma^2=\left(\alpha^2-\beta^2\right)+2j\alpha\beta \quad \textbf{(5)}$$

$$\nabla\times\overline{H}=\overline{J}+\frac{\partial\overline{D}}{\partial t}=\left(\sigma+j\omega\varepsilon\right)\overline{E} \quad \textbf{(1)}$$

$$\nabla\times\overline{E}=-\frac{\partial\overline{B}}{\partial t}=-j\omega\mu\overline{H} \quad \textbf{(2)}$$

$$\nabla\times\nabla\times\overline{E}=-j\omega\mu\left(\nabla\times\overline{H}\right)$$

Equating (4) and (5) for Re and Im parts :

$$\alpha^2-\beta^2=-\omega^2\mu\varepsilon \quad \text{(Re)} \quad (6)$$

$$2\alpha\beta=\omega\mu\sigma \quad \text{(Im)} \quad (7)$$

Magnitude for (5) ;

$$\left|\gamma^2\right| = \alpha^2 + \beta^2 \quad (8)$$

Magnitude for (4) ;

$$\left|\gamma^2\right| = \sqrt{\left(-\omega^2\mu\varepsilon\right)^2 + \left(\omega\sigma\mu\right)^2}$$
$$= \omega\mu\sqrt{\omega^2\varepsilon + \sigma^2} \quad (9)$$

Equate (8) and (9) :

$$\alpha^2 + \beta^2 = \omega\mu\sqrt{\omega^2\varepsilon + \sigma^2} \quad (10)$$

$$\alpha^2 - \beta^2 = -\omega^2\mu\varepsilon \quad (\mathrm{Re}) \quad (6)$$

Add (10) and (6) :

Hence :

$$2\alpha^2 = \omega\mu\sqrt{\omega^2\varepsilon + \sigma^2} - \omega^2\mu\varepsilon$$
$$= \omega^2\mu\varepsilon\sqrt{1 + \frac{\sigma^2}{\omega^2\varepsilon^2}} - \omega^2\mu\varepsilon$$

$$\alpha^2 = \frac{\omega^2\mu\varepsilon}{2}\left[\sqrt{1 + \frac{\sigma^2}{\omega^2\varepsilon^2}} - 1\right]$$

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1 + \frac{\sigma^2}{\omega^2\varepsilon^2}} - 1\right]} \; Np/m \quad (11)$$

α is known as *attenuation constant* as a measure of the *wave is attenuated* while traveling in a medium.

Substract (10) and (6) :

$$2\beta^2 = \omega\mu\sqrt{\omega^2\varepsilon + \sigma^2} + \omega^2\mu\varepsilon$$

$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}}+1\right]} \quad rad/m \quad (12)$$

β ***is phase constant***

If the electric field propagate in +z direction and has component x, the equation of the wave is given by :

$$\overline{E}(z,t) = E_0 e^{-\alpha z}\cos(\omega t - \beta z)\hat{x} \qquad \textbf{(13)}$$

And the magnetic field :

$$\overline{H}(z,t) = H_0 e^{-\alpha z}\cos(\omega t - \beta z - \theta_\eta)\hat{y} \qquad \textbf{(14)}$$

where ; $$H_0 = \frac{E_0}{|\eta|} \quad \textbf{(15)}$$

$$\bar{E}(z,t) = E_0 e^{-\alpha z}\cos(\omega t - \beta z)\hat{x} \quad \textbf{(14)}$$

$$\bar{H}(z,t) = H_0 e^{-\alpha z}\cos(\omega t - \beta z - \theta_\eta)\hat{y} \quad \textbf{(15)}$$

Intrinsic impedance :

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = |\eta|\angle\,\theta_\eta = |\eta| e^{j\theta_\eta}, (\Omega) \quad \textbf{(16)}$$

where ;

$$|\eta| = \frac{\sqrt{\mu/\varepsilon}}{\left[1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]^{1/4}}, \quad \tan 2\theta_\eta = \frac{\sigma}{\omega\varepsilon}, \quad 0 \leq \theta_\eta \leq 45^0 \quad \textbf{(17)}$$

Conclusions that can be made for the wave propagating in lossy dielectrics material :

(i) E and H fields amplitude will be attenuated by $e^{-\alpha z}$

(ii) E leading H by θ_η

Wave velocity ;

$$u = \omega / \beta \; ; \; \lambda = 2\pi / \beta$$

$$|\eta| = \frac{\sqrt{\mu / \varepsilon}}{\left[1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]^{1/4}}, \quad \tan 2\theta_\eta = \frac{\sigma}{\omega\varepsilon}, \quad 0 \le \theta_\eta \le 45^0 \quad (17)$$

Loss tangent ;

$$\frac{|\bar{J}|}{|\bar{J}_d|} = \frac{\sigma\bar{E}}{|j\omega\varepsilon\bar{E}|} = \frac{\sigma}{\omega\varepsilon} = \tan\theta \qquad \textbf{(18)}$$

From (17) and (18)

$$\theta = 2\theta_\eta$$

Loss tangent values will determine types of media :

tan θ small (σ / $\omega\varepsilon$ < 0.1) – good dielectric – low loss

tan θ large (σ /$\omega\varepsilon$ > 10) - good conductor – high loss

Another factor that determined the characteristic of the media is operating frequency. A medium can be regarded as a good conductor at low frequency might be a good dielectric at higher frequency.

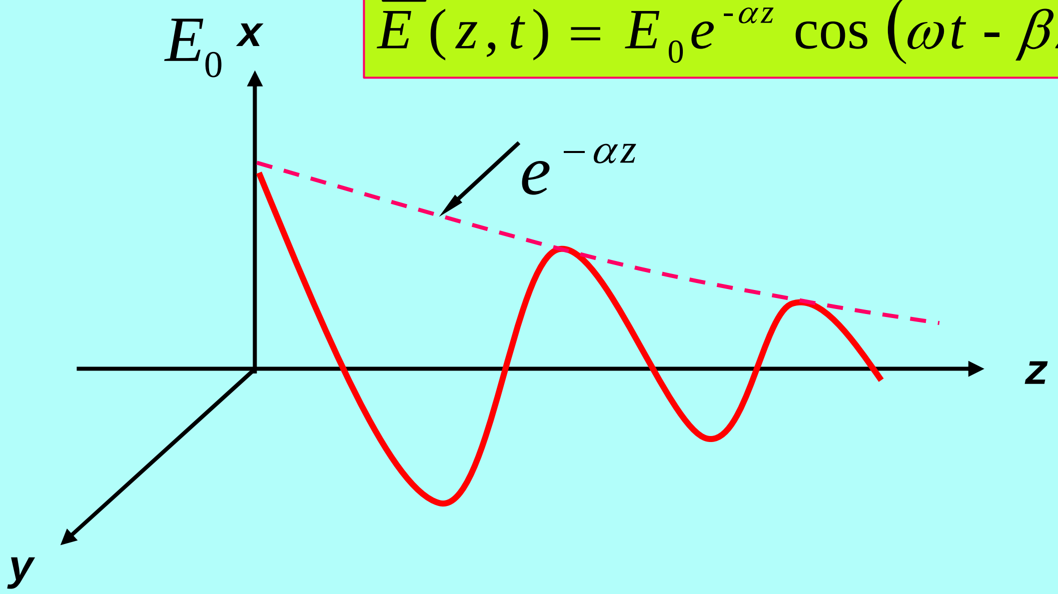
$$\overline{E}(z,t) = E_0 e^{-\alpha z}\cos(\omega t - \beta z)\hat{x} = E_0 e^{-\alpha z} e^{j(\omega t - \beta z)}\hat{x} \quad (14)$$

$$H_0 = \frac{E_0}{|\eta|}$$

$$\overline{H}(z,t) = H_0 e^{-\alpha z}\cos(\omega t - \beta z - \theta_\eta)\hat{y} = \frac{E_0}{|\eta|} e^{-\alpha z} e^{j(\omega t - \beta z - \theta_\eta)}\hat{y} \quad (15)$$

Graphical representation of E field in lossy dielectric

$$\overline{E}(z,t) = E_0 e^{-\alpha z}\cos(\omega t - \beta z)\hat{x}$$



10.3 PLANE WAVE IN LOSSLESS (PERFECT) DIELECTRICS

Characteristics: $\sigma = 0, \varepsilon = \varepsilon_0 \varepsilon_r, \mu = \mu_0 \mu_r$ **(19)**

Substitute in (11) and (12) :

$$\alpha = 0, \quad \beta = \omega\sqrt{\mu\varepsilon} \qquad \textbf{(20)}$$

$$u = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\varepsilon}}, \quad \lambda = \frac{2\pi}{\beta} \qquad \textbf{(21)}$$

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} \angle 0^o \qquad \textbf{(22)}$$

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}} - 1\right]} \; Np/m \quad (11)$$

$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}} + 1\right]} \; rad/m \quad (12)$$

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = |\eta| \angle \theta_\eta = |\eta| e^{j\theta_\eta}, (\Omega)$$

The zero angle means that E and H fields are in phase at each fixed location.

10.4 PLANE WAVE IN FREE SPACE

Free space is nothing more than the perfect dielectric media :

Characteristics: $\sigma = 0, \varepsilon = \varepsilon_0, \mu = \mu_0$ **(23)**

Substitute in (20) and (21) :

$$\alpha = 0 \ , \quad \beta = \omega\sqrt{\mu_0\varepsilon_0} = \omega / c \qquad (24)$$

$$u = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = c \ , \quad \lambda = \frac{2\pi}{\beta} \qquad (25)$$

where $u = c \approx 3\times10^8 \ m/s$

$$\eta = \eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 120\pi \ \ \Omega \qquad (26)$$

$$\alpha = 0, \quad \beta = \omega\sqrt{\mu\varepsilon} \qquad (20)$$

$$u = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\varepsilon}}, \quad \lambda = \frac{2\pi}{\beta} \qquad (21)$$

$$\eta = \sqrt{\frac{\mu}{\varepsilon}}\angle 0^o \qquad (22)$$

$$\mu = \mu_0 = 4\pi\times10^{-7} \ H/m$$

$$\varepsilon = \varepsilon_0 = 8.854\times10^{-12} \approx \frac{1}{36\pi}\times10^{-9} \ F/m$$

The field equations for E and H obtained :

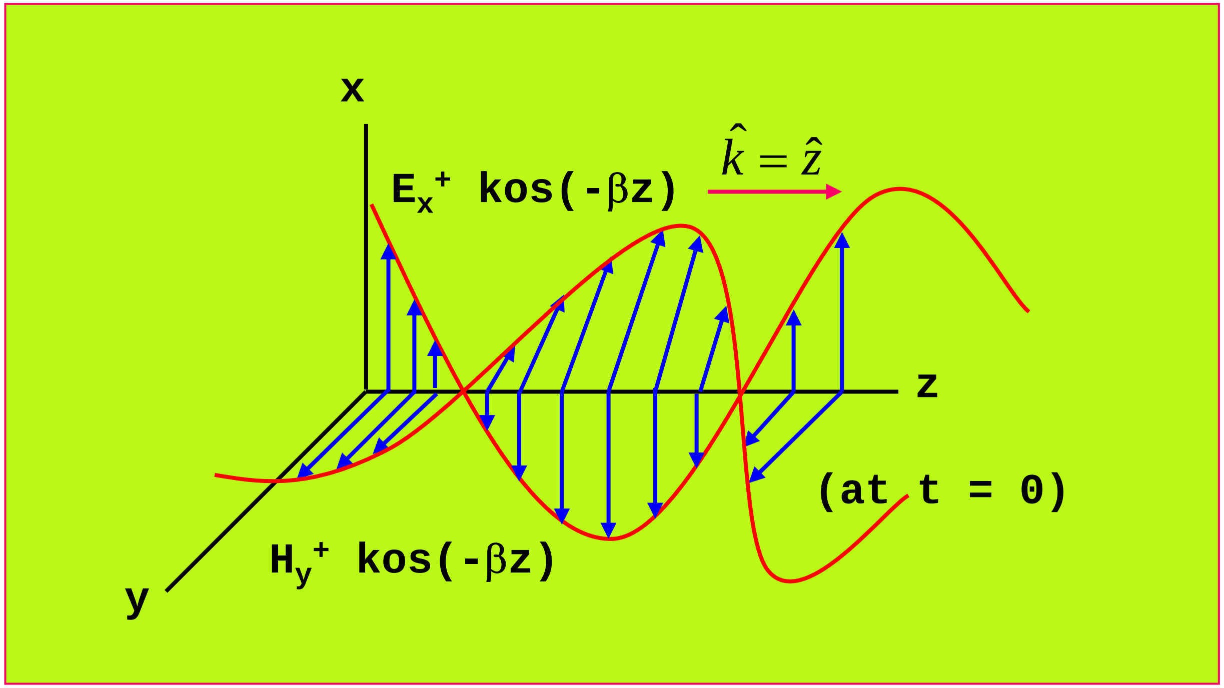
$$\overline{E} = E_0 \cos(\omega t - \beta z)\hat{x} \quad \textbf{(27)}$$

$$\overline{H} = \frac{E_0}{\eta_0}\cos(\omega t - \beta z)\hat{y} \quad \textbf{(28)}$$

$$\overline{E}(z,t) = E_0 e^{-\alpha z}\cos(\omega t - \beta z)\hat{x} \quad \textbf{(14)}$$

$$\overline{H}(z,t) = H_0 e^{-\alpha z}\cos(\omega t - \beta z - \theta_\eta)\hat{y} \quad \textbf{(15)}$$

E and H fields and the direction of propagation :



Generally :

$$\hat{E} \times \hat{H} = \hat{k}$$

10.5 PLANE WAVE IN CONDUCTORS

In conductors : $\sigma >> \omega\varepsilon$ *or* $\dfrac{\sigma}{\omega\varepsilon} \rightarrow \infty$

With the characteristics : $\sigma \sim> \infty, \varepsilon = \varepsilon_0, \mu = \mu_0\mu_r$ **(29)**

Substitute in (11 and (12) :

$$\alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\pi f \mu\sigma} \quad \textbf{(30)}$$

$$\eta = \sqrt{\frac{\omega\mu}{\sigma}}\angle 45^o$$

E leads H by 45⁰ **(31)**

The field equations for E and H obtained :

$$\overline{E} = E_0 e^{-\alpha z}\cos(\omega t - \beta z)\hat{x} \quad \textbf{(32)}$$

$$\overline{H} = \frac{E_0}{\eta_0} e^{-\alpha z}\cos(\omega t - \beta z - 45^o)\hat{y} \quad \textbf{(33)}$$

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}} - 1\right]} \; Np/m \; (11)$$

$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}} + 1\right]} \; rad/m \; (12)$$

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = |\eta|\angle\theta_\eta = |\eta|e^{j\theta_\eta}, (\Omega)$$

It is seen that in conductors $\overline{E}$ and $\overline{H}$ waves are attenuated by $e^{-\alpha z}$

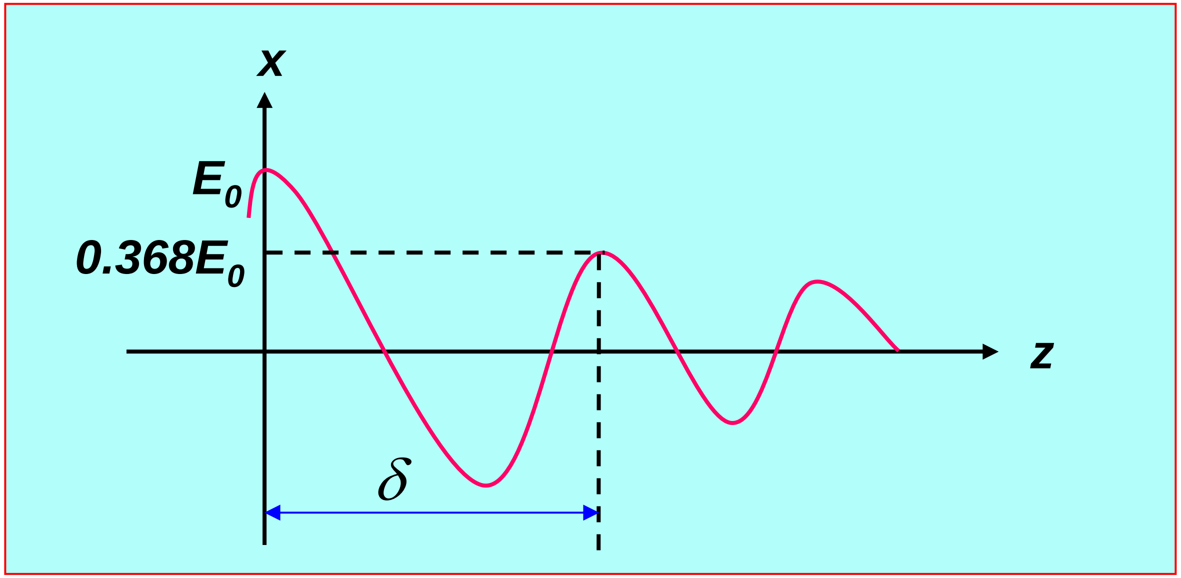
From the diagram δ is referred to as the skin depth. It refers to the amplitude of the wave propagate to a conducting media is reduced to e^{-1} or 37% from its initial value.

In a distance :

$$E_0 e^{-\alpha\delta} = E_0 e^{-1}$$

$$\therefore \ \delta = 1/\alpha = \frac{1}{\sqrt{\pi f \mu \sigma}} \qquad (34)$$

It can be seen that at higher frequencies δ is decreasing.



Ex.10.1 : ***A lossy dielectric has an intrinsic impedance of $200\angle 30^o\ \Omega$ at the particular frequency. If at that particular frequency a plane wave that propagate in a medium has a magnetic field given by :***

$$\overline{H} = 10e^{-\alpha x}\cos(\omega t - x/2)\,\hat{y}\ A/m.$$

Find $\overline{E}$ and α .

Solution :

$$\hat{E}\times\hat{H} = \hat{k}$$

$$\rightarrow\ \hat{E}\times\hat{y} = \hat{x}$$

$$\therefore\ \overline{E} = -\hat{z}$$

From intrinsic impedance, the magnitude of E field :

$$\eta = \frac{E_0}{H_0} = 200\angle 30^o$$

$$\rightarrow\ E_0 = 2000\angle 30^o$$

It is seen that E field leads H field :

$$\theta_\eta = 30^0 = \pi/6$$

Hence :

$$\overline{E} = -2000e^{-\alpha x}\cos(\omega t - x/2 + \pi/6)\,\hat{z}\ \ (V/m)$$

$$\overline{E} = -2000\, e^{-\alpha x} \cos(\omega t - x/2 + \pi/6)\,\hat{z} \quad (V/m)$$

To find α:

$$\frac{\alpha}{\beta} = \left[\frac{\sqrt{1 + \left(\sigma / \omega\varepsilon \right)^2} - 1}{\sqrt{1 + \left(\sigma / \omega\varepsilon \right)^2} + 1} \right]^{1/2}$$

$$\sigma / \omega\varepsilon = \tan 2\theta_\eta = \tan 60^0 = \sqrt{3}$$

$$\therefore \frac{\alpha}{\beta} = \left[\frac{2-1}{2+1} \right]^{1/2} = \frac{1}{\sqrt{3}}$$ ***; and we know*** $\beta = 1/2$

$$\rightarrow \alpha = \frac{\beta}{\sqrt{3}} = 0.2887\ Np/m$$

Hence:

$$\overline{E} = -2000\, e^{-0.2887\, x} \cos(\omega t - x/2 + \pi/6)\,\hat{z} \quad (V/m)$$

10.6 POWER AND THE POYNTING VECTOR

$$\nabla \times \overline{E} = -\mu \frac{\partial \overline{H}}{\partial t} \qquad \textbf{(35)}$$

$$\nabla \times \overline{H} = \sigma \overline{E} + \varepsilon \frac{\partial \overline{E}}{\partial t} \qquad \textbf{(36)}$$

Dot product (36) with $\overline{E}$ ***:***

$$\overline{E} \bullet \left(\nabla \times \overline{H}\right) = \sigma E^2 + \overline{E} \bullet \varepsilon \frac{\partial \overline{E}}{\partial t} \qquad \textbf{(37)}$$

From vector identity:

$$\nabla \bullet \left(\overline{A} \times \overline{B}\right) = \overline{B} \bullet \left(\nabla \times \overline{A}\right) - \overline{A} \bullet \left(\nabla \times \overline{B}\right) \qquad \textbf{(38)}$$

Change $\overline{A} = \overline{H}, \overline{B} = \overline{E}$ ***in (37) and use (38) , equation (37) becomes :***

$$\overline{H} \bullet \left(\nabla \times \overline{E}\right) + \nabla \bullet \left(\overline{H} \times \overline{E}\right) = \sigma E^2 + \overline{E} \bullet \varepsilon \frac{\partial \overline{E}}{\partial t} \qquad \textbf{(39)}$$

$$\overline{H} \bullet \left(\nabla \times \overline{E}\right) + \nabla \bullet \left(\overline{H} \times \overline{E}\right) = \sigma E^2 + \overline{E} \bullet \varepsilon \frac{\partial \overline{E}}{\partial t} \quad \textbf{(39)}$$

$$\nabla \times \overline{E} = -\mu \frac{\partial \overline{H}}{\partial t} \quad \textbf{(35)}$$

And from (35):

$$\overline{H} \bullet \left(\nabla \times \overline{E}\right) = \overline{H} \bullet \left(-\mu \frac{\partial \overline{H}}{\partial t} \right) = -\frac{\mu}{2} \frac{\partial}{\partial t} \overline{H} \bullet \overline{H} \quad \textbf{(40)}$$

Therefore (39) becomes:

$$-\frac{\mu}{2} \frac{\partial H^2}{\partial t} - \nabla \bullet \left(\overline{E} \times \overline{H}\right) = \sigma \overline{E}^2 + \overline{E} \bullet \varepsilon \frac{\partial \overline{E}}{\partial t} \quad \textbf{(41)}$$

where:

$$\nabla \bullet \left(\overline{H} \times \overline{E}\right) = -\nabla \bullet \left(\overline{E} \times \overline{H}\right)$$

Integration (41) throughout volume v :

$$\int_v \nabla \bullet \left(\overline{E} \times \overline{H}\right) dv = -\frac{\partial}{\partial t} \int_v \left[\frac{1}{2} \varepsilon E^2 + \frac{1}{2} \mu H^2 \right] dv - \int_v \sigma E^2 dv \quad \textbf{(42)}$$

$$\int_v \nabla \bullet \left(\overline{E} \times \overline{H}\right) dv = -\frac{\partial}{\partial t} \int_v \left[\frac{1}{2}\varepsilon E^2 + \frac{1}{2}\mu H^2\right] dv - \int_v \sigma E^2 dv \quad \textbf{(42)}$$

Using divergence theorem to (42):

$$\oint_s \left(\overline{E} \times \overline{H}\right) \bullet d\overline{S} = -\frac{\partial}{\partial t} \int_v \left[\frac{1}{2}\varepsilon E^2 + \frac{1}{2}\mu H^2\right] dv - \int_v \sigma E^2 dv \quad (43)$$

Total energy flow leaving the volume

The decrease of the energy densities of energy stored in the electric and magnetic fields

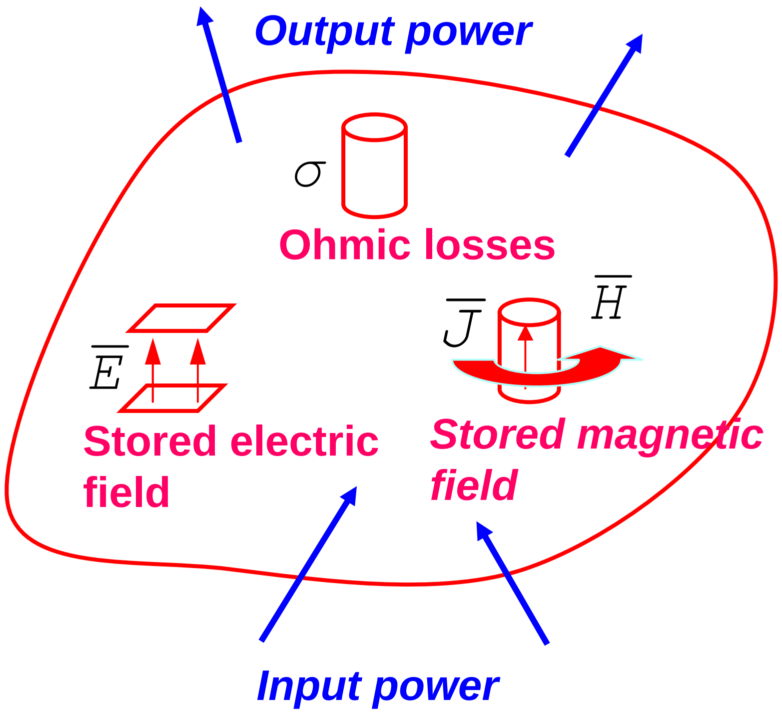
Dissipated ohmic power

Equation (43) shows Poynting Theorem and can be written as :

$$\wp = \overline{E} \times \overline{H} \quad W/m^2$$

Poynting theorem* states that the *total power flow leaving the volume* is equal to *the decrease of the energy densities of energy stored in the electric and magnetic fields* and *the dissipated ohmic power.

The theorem can be explained as shown in the diagram below :



Given for lossless dielectric, the electric and magnetic fields are :

$$\overline{E} = E_0 \cos(\omega t - \beta z)\hat{x}$$

$$\overline{H} = \frac{E_0}{\eta}\cos(\omega t - \beta z)\hat{y}$$

The Poynting vector becomes:

$$\wp = \overline{E} \times \overline{H} \quad W/m^2$$

$$\wp = \frac{E^2{}_0}{\eta}\cos^2(\omega t - \beta z)\hat{y}$$

To find average power density :

Integrate Poynting vector and divide with interval T = 1/f :

$$P_{ave} = \frac{1}{T}\int_0^T \frac{E_0^2}{\eta}\cos^2(\omega t - \beta z)\,dt$$

$$= \frac{1}{2T}\frac{E_0^2}{\eta}\int_0^T \left[1 + \cos(2\omega t - 2\beta z)\right]dt$$

$$= \frac{1}{2T}\frac{E_0^2}{\eta}\left[t + \frac{1}{2\omega}\sin(2\omega t - 2\beta z)\Big|_0^T\right]$$

$$\therefore P_{ave} = \frac{1}{2}\frac{E_0^2}{\eta} \quad W/m^2$$

Average power through area S :

$$P_{ave} = \frac{1}{2}\frac{E_0^2}{\eta} S \quad (W)$$

Given for lossy dielectric, the electric and magnetic fields are :

$$\overline{E} = E_0 e^{-\alpha z} \cos(\omega t - \beta z)\hat{x}$$

$$\overline{H} = \frac{E_0}{\eta_0} e^{-\alpha z} \cos(\omega t - \beta z - \theta_\eta)\hat{y}$$

The Poynting vector becomes:

$$\wp = \frac{E^2{}_0}{\eta} e^{-2\alpha z} \cos(\omega t - \beta z)\cos(\omega t - \beta z - \theta_\eta)$$

Average power :

$$P_{ave} = \frac{1}{2}\frac{E_0^2}{\eta} e^{-2\alpha z} \cos\theta_\eta$$

Ex.10.2: ***A uniform plane wave propagate in a lossless dielectric in the +z direction. The electric field is given by :***

$$\overline{E}(z,t) = 377\cos\left(\omega t - (4\pi/3)z + \pi/6\right)\hat{x}\ (V/m)$$

The average power density measured was $377\ W/m^2$. ***Find:***

(i) Dielectric constant of the material if $\mu = \mu_0$

(ii) Wave frequency

(iii) Magnetic field equation

Solution:

(i) Average power :

$$P_{ave} = \frac{1}{2}\frac{E^2}{\eta} = 377$$

$$\frac{1}{2}\frac{(377)^2}{\eta} = 377$$

$$\rightarrow \eta = 377/2 = 188.5\Omega$$

For lossless dielectric :

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_0}{\varepsilon_r \varepsilon_0}}$$

$$\sqrt{\varepsilon_r} = \frac{1}{\eta}\sqrt{\frac{\mu_0}{\varepsilon_0}} = 1.9986$$

$$\rightarrow \varepsilon_r = 4.0$$

$$\overline{E}(z,t) = 377\cos\left(\omega t - \left(4\pi/3\right)z + \pi/6\right)\hat{x}\,(V/m)$$

(ii) Wave frequency :

$$\beta = 4\pi/3 = \omega\sqrt{\mu_0 \varepsilon}$$

$$\omega = \frac{4\pi}{3\sqrt{\mu_0 \varepsilon}}$$

$$2\pi f = 3.9946 \times 10^{16}$$

$$\rightarrow f = 99.93 \times 10^{6} \approx (100 MHz)$$

$$\overline{E}(z,t) = 377\ \cos\left(\omega t - \left(4\pi/3\right)z + \pi/6\right)\hat{x}\ (V/m)$$

(iii) Magnetic field equation :

$$\overline{H}(z,t) = \frac{377}{\eta}\cos(\omega t - (4\pi/3)z + \pi/6)\,\hat{y}$$

$$= 2\cos(\omega t - (4\pi/3)z + \pi/6)\,\hat{y}\ (A/m)$$